Canonicity of Proofs in Constructive Modal Logic

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VILLUM FONDEN

“X-IDF: Explainable Internet Data Flows”

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Joint work with

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Previously…

Propositional Modal Formulas

$$A, B ::= a \mid A \supset B \mid A \wedge B \mid \Box A \mid \Diamond A \mid 1$$

Constructive Modal Logic (CK)

=

Intuitionistic propositional logic (LI)

+

Nec rule: if F is provable, then $\Box F$ is provable

+

$\mathsf{k}_1 : \Box(A \supset B) \supset (\Box A \supset \Box B)$ $\mathsf{k}_2 : \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$

$$\frac{}{a \vdash a}\,\mathsf{AX} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,\supset^{\mathsf{R}} \quad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\,\supset^{\mathsf{L}} \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge^{\mathsf{R}} \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge^{\mathsf{L}}$$

$$\frac{}{\vdash 1}\,1 \quad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathsf{C} \quad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathsf{W} \quad \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\mathsf{K}^{\Box} \quad \frac{A, \Gamma \vdash B}{\Diamond A, \Box\Gamma \vdash \Diamond B}\,\mathsf{K}^{\Diamond}$$

Lambda calculus for (constructive) modal logic:

- Pfenning [1995,2001]
- Bellin, De Paiva and Ritter [2001]
- Kakutani [2007]
- Kavvos [2017]

$$M, N \coloneqq x \mid \lambda x.M \mid (MN) \mid \text{Let } x_1, \ldots x_n \text{ be } N_1, \ldots, N_n \text{ in } M$$

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$$M, N \coloneqq x \mid \lambda x.M \mid (MN) \mid M\,[N_1, \ldots, N_n / x_1, \ldots x_n]_{\blacksquare}$$

$$(\lambda x.M)N \rightsquigarrow_\beta M\,\{N/x\}$$

$$M\left[\vec{P}, R\left[\vec{N}/\vec{z}\right]_{\blacksquare}, \vec{Q}/\vec{x}, y, \vec{w}\right]_{\blacksquare} \rightsquigarrow_\beta M\,\{R/y\}\left[\vec{P}, \vec{N}, \vec{Q}/\vec{x}, \vec{z}, \vec{w}\right]_{\blacksquare}$$

Note: no η-expansion of the $\cdot\,[\cdot/\cdot]_{\blacksquare}$

What about semantics*?

What about semantics*?
*not those semantics

What about denotational semantics?

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Definition (Denotational semantics)

$$\begin{array}{rccc} [\![-]\!]: & \{\text{ Proofs }\} & \to & \{\text{ Denotations }\} \\ & \mathfrak{D} & \to & [\![\mathfrak{D}]\!] \end{array}$$

such that if $\mathfrak{D} \rightsquigarrow^*_{\text{cut}} \mathfrak{D}'$ then $[\![\mathfrak{D}]\!] = [\![\mathfrak{D}']\!]$

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Denotational semantics for constructive modal logic(s)
[Bellin, De Paiva and Ritter, 2001]

Idea: $\frac{\{\lambda\text{-terms}\}}{\beta\text{-reduction}} \left(\text{which is tha same of} \frac{\{\text{Proofs}\}}{\text{cut-elimination}}\right)$

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Is it possible to have a concrete model?
[i.e., denotations are not equivalence classes]

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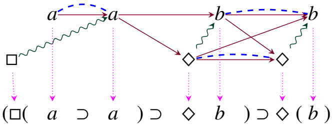
Is it possible to have a concrete model?
[i.e., denotations are not equivalence classes]
What about game semantics (à la Blass, Abramsky, Hyland)?

- AiML 2022 [A & L. Straßburger]:
  Combinatorial Proofs for Constructive Modal Logic

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a}\,\mathsf{AX}}{\vdash a \supset a}\,{\supset}^{\mathsf{R}}}{\vdash \Box(a \supset a)}\,\mathsf{K}^{\Box} \qquad \mathsf{K}^{\Diamond}\,\dfrac{\dfrac{}{b \vdash b}\,\mathsf{AX}}{\Diamond b \vdash \Diamond b}}{\dfrac{\Box(a \supset a) \supset \Diamond b \vdash \Diamond b}{((\Box a \supset a) \supset \Diamond b) \supset \Diamond b}\,{\supset}^{\mathsf{R}}}\,{\supset}^{\mathsf{L}}
\quad \leftrightsquigarrow
$$



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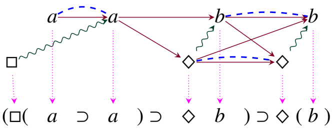
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a}\,\mathsf{AX}}{\vdash a \supset a}\supset^{\mathsf{R}}}{\vdash \Box(a \supset a)}\,\mathsf{K}^{\Box} \qquad \dfrac{\dfrac{}{b \vdash b}\,\mathsf{AX}}{\Diamond b \vdash \Diamond b}\,\mathsf{K}^{\Diamond}}{\Box(a \supset a) \supset \Diamond b \vdash \Diamond b}\supset^{\mathsf{L}}}{((\Box a \supset a) \supset \Diamond b) \supset \Diamond b}\supset^{\mathsf{R}} \quad \rightsquigarrow$$



- TABLEAUX 2021 [A. & D. Catta & L. Straßburger]:
  Games Semantics for Constructive Modal Logic

$$\left[\!\!\left[\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a}\,\mathsf{AX}}{\vdash a \supset a}\supset^{\mathsf{R}}}{\vdash \Box(a \supset a)}\,\mathsf{K}^{\Box} \qquad \dfrac{\dfrac{}{b \vdash b}\,\mathsf{AX}}{\Diamond b \vdash \Diamond b}\,\mathsf{K}^{\Diamond}}{\Box(a \supset a) \supset \Diamond b \vdash \Diamond b}\supset^{\mathsf{L}}}{((\Box a \supset a) \supset \Diamond b) \supset \Diamond b}\supset^{\mathsf{R}}\right]\!\!\right] \to \left\{ \epsilon \quad \begin{array}{cccc} b_0, & b_0 b_1, & b_0 b_1 a_0, & b_0 b_1 a_0 a_1 \\ \Diamond_0, & \Diamond_0 \Diamond_1, & \Diamond_0 \Diamond_1 a_0, & \Diamond_0 \Diamond_1 a_0 a_1 \end{array} \right\}$$

Even previously…

A very minimal fragment of intuitionistic Logic

$$A, B ::= 1 \mid a \mid A \supset B$$

Sequent Calulus

$$\frac{}{a \vdash a}\,\mathsf{AX} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\supset^{\mathsf{R}} \quad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\supset^{\mathsf{L}} \quad \frac{}{\vdash 1}\,1 \quad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathsf{C} \quad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathsf{W}$$

Theorem

One-to-one correspondence between $\beta\eta$-normal λ-terms and WIS*s.*

Terms	*Reductions*
$t := \star \mid x \mid \lambda x.t \mid (t)u$	$(\lambda x.t)u \rightsquigarrow_\beta t\{u/x\} \qquad t \rightsquigarrow_\eta \lambda x.t(x)$

Theorem (Denotational Semantics)

WIS*s provide a full complete denotational semantics for intuitionistic logic.*

- *If $\mathcal{S}$ is a* WIS*, then there is π s.t. $\mathcal{S} = [\![\pi]\!]$*
- $\pi_1 \rightsquigarrow \hat{\pi} \leftsquigarrow \pi_2 \iff [\![\pi_1]\!] = [\![\pi_2]\!]$

Previously…

I mean the first “Previously”,

not “Previously the even previously”

Independent rules	$\dfrac{\Gamma_1,\Delta_1 \qquad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \qquad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$ $\dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1$ $\dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$
Resource Management	$\dfrac{\dfrac{\Gamma,A,A,B,B \vdash C}{\Gamma,A,B \vdash C}2\times\mathsf{C}}{\Gamma,A\wedge B \vdash C}\wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\dfrac{\Gamma,A,A,B,B \vdash C}{\Gamma,A\wedge B,A\wedge B \vdash C}2\times\wedge^{\mathsf{L}}}{\Gamma,A\wedge B \vdash C}\mathsf{C}$ $\dfrac{\dfrac{\Gamma \vdash C}{\Gamma,A,B \vdash C}2\times\mathsf{W}}{\Gamma,A\wedge B \vdash C}\wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\Gamma \vdash C}{\Gamma,A\wedge B \vdash C}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,A,A \vdash B}{\Gamma,A \vdash B}\mathsf{C}}{\Gamma,A,A \vdash B}\mathsf{W} \equiv_{\mathsf{c}} \Gamma,A,A \vdash B$ $\dfrac{\dfrac{\Gamma,A \vdash B}{\Gamma,A,A \vdash B}\mathsf{W}}{\Gamma,A \vdash B}\mathsf{C} \equiv_{\mathsf{c}} \Gamma,A \vdash B$
Excising and Unfolding	$\dfrac{\Gamma \vdash A \qquad \dfrac{\Delta \vdash C}{B,\Delta \vdash C}\mathsf{W}}{\Gamma,\Delta,A\supset B \vdash C}\supset^{\mathsf{L}} \equiv_{\mathsf{e}} \dfrac{\Delta \vdash C}{\Gamma,\Delta,A\supset B \vdash C}\mathsf{W}$ $\dfrac{\Gamma \vdash A \qquad \dfrac{\Delta,B,B \vdash C}{\Delta,B \vdash C}\mathsf{C}}{\Gamma,A\supset B \vdash C}\supset^{\mathsf{L}} \equiv_{\mathsf{u}} \dfrac{\dfrac{\Gamma \vdash A \qquad \dfrac{\Gamma \vdash A \quad \Delta,B,B \vdash C}{\Gamma,\Delta,A\supset B,B \vdash C}\supset^{\mathsf{L}}}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B \vdash C}\supset^{\mathsf{L}}}{\Gamma,\Delta,A\supset B \vdash C}\mathsf{C}$
Structural vs K	$\dfrac{\dfrac{\Gamma \vdash A}{\Gamma,B \vdash A}\mathsf{W}}{\Box\Gamma,\Box B \vdash \Box A}\mathsf{K}^{\Box} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\mathsf{K}^{\Box}}{\Box\Gamma,\Box B \vdash \Box A}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,B,B \vdash A}{\Gamma,B \vdash A}\mathsf{C}}{\Box\Gamma,\Box B \vdash \Box A}\mathsf{K}^{\Box} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B,B \vdash A}{\Box\Gamma,\Box B,\Box B \vdash \Box A}\mathsf{K}^{\Box}}{\Box\Gamma,\Box B \vdash \Box A}\mathsf{C}$ $\dfrac{\dfrac{\Gamma,B \vdash A}{\Gamma,B,C \vdash A}\mathsf{W}}{\Box\Gamma,\Diamond B,\Box C, \vdash \Box A}\mathsf{K}^{\Diamond} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B \vdash A}{\Box\Gamma,\Diamond B \vdash \Diamond A}\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Box C \vdash \Box A}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,B,C,C \vdash A}{\Gamma,B,C \vdash A}\mathsf{C}}{\Box\Gamma,\Diamond B,\Box C \vdash \Diamond A}\mathsf{K}^{\Box} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B,C,C \vdash A}{\Box\Gamma,\Diamond B,\Box C,\Box C\Box \vdash \Diamond A}\mathsf{K}^{\Box}}{\Box\Gamma,\Diamond B,\Box C \vdash \Diamond A}\mathsf{C}$
Jumps	$\dfrac{\dfrac{\dfrac{\Gamma \vdash A}{\Gamma,B \vdash A}\mathsf{W}}{\Box\Gamma,\Diamond B \vdash \Diamond A}\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Diamond C \vdash \Diamond A}\mathsf{W} \equiv_{\Diamond\mathsf{w}} \dfrac{\dfrac{\dfrac{\Gamma \vdash A}{\Gamma,C \vdash A}\mathsf{W}}{\Box\Gamma,\Diamond C \vdash \Diamond A}\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Diamond C \vdash \Diamond A}\mathsf{W}$

Independent rules	$\dfrac{\Gamma_1, \Delta_1 \quad \dfrac{\Gamma_2, \Delta_2, \Delta_3 \quad \Gamma_3, \Delta_4}{\Gamma_2, \Gamma_3, \Delta_2, \Sigma_2}\rho_2}{\Gamma_1, \Gamma_2, \Gamma_3, \Sigma_1, \Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma_1, \Delta_1 \quad \Gamma_1, \Delta_2, \Delta_3}{\Gamma_1, \Gamma_2, \Sigma_1, \Delta_2,}\rho_1 \quad \Gamma_3, \Delta_4}{\Gamma_1, \Gamma_2, \Gamma_3, \Sigma_1, \Sigma_2}\rho_2$ $\quad \dfrac{\dfrac{\Gamma, \Delta_1, \Delta_2}{\Gamma, \Sigma_1, \Delta_2}\rho_1}{\Gamma, \Sigma_1, \Sigma_2}\rho_2 \equiv \dfrac{\dfrac{\Gamma, \Delta_1, \Delta_2}{\Gamma, \Delta_1, \Sigma_2}\rho_2}{\Gamma, \Sigma_1, \Sigma_2}\rho_1$ $\quad \dfrac{\dfrac{\Gamma, \Delta_1, \Delta_2 \quad \Gamma_2, \Delta_3}{\Gamma_1, \Gamma_2, \Delta_1, \Sigma_2}\rho_2}{\Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma, \Delta_1, \Delta_2}{\Gamma, \Sigma_1, \Delta_2}\rho_1 \quad \Gamma_2, \Delta_3}{\Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2}\rho_2$
Resource Management	$\dfrac{\dfrac{\Gamma, A, A, B, B \vdash C}{\Gamma, A, B \vdash C} 2 \times \mathsf{C}}{\Gamma, A \wedge B \vdash C} \wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\dfrac{\Gamma, A, A, B, B \vdash C}{\Gamma, A \wedge B, A \wedge B \vdash C} 2 \times \wedge^{\mathsf{L}}}{\Gamma, A \wedge B \vdash C} \mathsf{C}$ $\quad \dfrac{\dfrac{\Gamma \vdash C}{\Gamma, A, B \vdash C} 2 \times \mathsf{W}}{\Gamma, A \wedge B \vdash C} \wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\Gamma \vdash C}{\Gamma, A \wedge B \vdash C} \mathsf{W}$ $\dfrac{\dfrac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \mathsf{C}}{\Gamma, A, A \vdash B} \mathsf{W} \equiv_{\mathsf{c}} \Gamma, A, A \vdash B$ $\quad \dfrac{\dfrac{\Gamma, A \vdash B}{\Gamma, A, A \vdash B} \mathsf{W}}{\Gamma, A \vdash B} \mathsf{C} \equiv_{\mathsf{c}} \Gamma, A \vdash B$
Excising and Unfolding	$\dfrac{\Gamma \vdash A \quad \dfrac{\Delta \vdash C}{B, \Delta \vdash C} \mathsf{W}}{\Gamma, \Delta, A \supset B \vdash C} \supset^{\mathsf{L}} \equiv_{\mathsf{e}} \dfrac{\Delta \vdash C}{\Gamma, \Delta, A \supset B \vdash C} \mathsf{W}$
Structural vs K	
Jumps	

$$\equiv_{\mathsf{CP}} := (\equiv \cup \equiv_{\mathsf{c}} \cup \equiv_{\mathsf{e}})$$

Independent rules	$\dfrac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$ $\quad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1$ $\quad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$
Resource Management	$\dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A,B\vdash C}2\times C}{\Gamma,A\wedge B\vdash C}\wedge^L \equiv_c \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}2\times\wedge^L}{\Gamma,A\wedge B\vdash C}C$ $\quad \dfrac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}2\times W}{\Gamma,A\wedge B\vdash C}\wedge^L \equiv_c \dfrac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}W$ $\dfrac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}C}{\Gamma,A,A\vdash B}W \equiv_c \Gamma,A,A\vdash B$ $\quad \dfrac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}W}{\Gamma,A\vdash B}C \equiv_c \Gamma,A\vdash B$
Excising and Unfolding	$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta\vdash C}{B,\Delta\vdash C}W}{\Gamma,\Delta,A\supset B\vdash C}\supset^L \equiv_e \dfrac{\Delta\vdash C}{\Gamma,\Delta,A\supset B\vdash C}W$ $\quad \Vert \quad \dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}C}{\Gamma,A\supset B\vdash C}\supset^L \equiv_u \dfrac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\supset^L}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\supset^L}{\Gamma,\Delta,A\supset B\vdash C}C$
Structural vs K	$\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}W}{\Box\Gamma,\Box B\vdash\Box A}K^\Box \equiv_{\Box c} \dfrac{\dfrac{\Gamma\vdash A}{\Box\Gamma\vdash\Box A}K^\Box}{\Box\Gamma,\Box B\vdash\Box A}W$ $\quad \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Gamma,B\vdash A}C}{\Box\Gamma,\Box B\vdash\Box A}K^\Box \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Box\Gamma,\Box B,\Box B\vdash\Box A}K^\Box}{\Box\Gamma,\Box B\vdash\Box A}C$ $\dfrac{\dfrac{\Gamma,B\vdash A}{\Gamma,B,C\vdash A}W}{\Box\Gamma,\Diamond B,\Box C,\vdash\Box A}K^\Diamond \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B\vdash A}{\Box\Gamma,\Diamond B\vdash\Diamond A}K^\Diamond}{\Box\Gamma,\Diamond B,\Box C\vdash\Box A}W$ $\quad \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Gamma,B,C\vdash A}C}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}K^\Box \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Box\Gamma,\Diamond B,\Box C,\Box C\Box\vdash\Diamond A}K^\Box}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}C$
Jumps	

$$\equiv_{\mathsf{WIS}} := (\equiv_\lambda \cup \equiv_{\Box c})$$

Independent rules	$\dfrac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$ $\quad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1$ $\quad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$
Resource Management	$\dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A,B\vdash C}2\times C}{\Gamma,A\wedge B\vdash C}\wedge^L \equiv_c \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}2\times\wedge^L}{\Gamma,A\wedge B\vdash C}C$ $\quad \dfrac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}2\times W}{\Gamma,A\wedge B\vdash C}\wedge^L \equiv_c \dfrac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}W$ $\dfrac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}C}{\Gamma,A,A\vdash B}W \equiv_c \Gamma,A,A\vdash B$ $\quad \dfrac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}W}{\Gamma,A\vdash B}C \equiv_c \Gamma,A\vdash B$
Excising and Unfolding	$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta\vdash C}{B,\Delta\vdash C}W}{\Gamma,\Delta,A\supset B\vdash C}\supset^L \equiv_e \dfrac{\Delta\vdash C}{\Gamma,\Delta,A\supset B\vdash C}W$ $\quad \Vert \quad \dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}C}{\Gamma,A\supset B\vdash C}\supset^L \equiv_u \dfrac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\supset^L}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\supset^L}{\Gamma,\Delta,A\supset B\vdash C}C$
Structural vs K	$\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}W}{\Box\Gamma,\Box B\vdash\Box A}K^\Box \equiv_{\Box c} \dfrac{\dfrac{\Gamma\vdash A}{\Box\Gamma\vdash\Box A}K^\Box}{\Box\Gamma,\Box B\vdash\Box A}W$ $\quad \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Gamma,B\vdash A}C}{\Box\Gamma,\Box B\vdash\Box A}K^\Box \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Box\Gamma,\Box B,\Box B\vdash\Box A}K^\Box}{\Box\Gamma,\Box B\vdash\Box A}C$ $\dfrac{\dfrac{\Gamma,B\vdash A}{\Gamma,B,C\vdash A}W}{\Box\Gamma,\Diamond B,\Box C,\vdash\Box A}K^\Diamond \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B\vdash A}{\Box\Gamma,\Diamond B\vdash\Diamond A}K^\Diamond}{\Box\Gamma,\Diamond B,\Box C\vdash\Box A}W$ $\quad \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Gamma,B,C\vdash A}C}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}K^\Box \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Box\Gamma,\Diamond B,\Box C,\Box C\Box\vdash\Diamond A}K^\Box}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}C$
Jumps	$\dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}W}{\Box\Gamma,\Diamond B\vdash\Diamond A}K^\Diamond}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}W \equiv_{\Diamond w} \dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,C\vdash A}W}{\Box\Gamma,\Diamond C\vdash\Diamond A}K^\Diamond}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}W$

$$\equiv_{\Diamond w} := (\equiv_{WIS}\cup\equiv_{\Box c})$$

Independent rules	$\dfrac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$ $\dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1$ $\dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$
Resource Management	$\dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A,B\vdash C}2\times\mathsf{C}}{\Gamma,A\wedge B\vdash C}\wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}2\times\wedge^{\mathsf{L}}}{\Gamma,A\wedge B\vdash C}\mathsf{C}$ $\dfrac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}2\times\mathsf{W}}{\Gamma,A\wedge B\vdash C}\wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}\mathsf{C}}{\Gamma,A,A\vdash B}\mathsf{W} \equiv_{\mathsf{c}} \Gamma,A,A\vdash B$ $\dfrac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}\mathsf{W}}{\Gamma,A\vdash B}\mathsf{C} \equiv_{\mathsf{c}} \Gamma,A\vdash B$
Excising and Unfolding	$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta\vdash C}{B,\Delta\vdash C}\mathsf{W}}{\Gamma,\Delta,A\supset B\vdash C}\supset^{\mathsf{L}} \equiv_{\mathsf{e}} \dfrac{\Delta\vdash C}{\Gamma,\Delta,A\supset B\vdash C}\mathsf{W}$ $\dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}\mathsf{C}}{\Gamma,A\supset B\vdash C}\supset^{\mathsf{L}} \equiv_{\mathsf{u}} \dfrac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\supset^{\mathsf{L}}}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\supset^{\mathsf{L}}}{\Gamma,\Delta,A\supset B\vdash C}\mathsf{C}$
Structural vs K	$\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}\mathsf{W}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{K}^{\Box} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma\vdash A}{\Box\Gamma\vdash\Box A}\mathsf{K}^{\Box}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,B,B\vdash A}{\Gamma,B\vdash A}\mathsf{C}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{K}^{\Box} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Box\Gamma,\Box B,\Box B\vdash\Box A}\mathsf{K}^{\Box}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{C}$ $\dfrac{\dfrac{\Gamma,B\vdash A}{\Gamma,B,C\vdash A}\mathsf{W}}{\Box\Gamma,\Diamond B,\Box C,\vdash\Box A}\mathsf{K}^{\Diamond} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B\vdash A}{\Box\Gamma,\Diamond B\vdash\Diamond A}\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Box C\vdash\Box A}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Gamma,B,C\vdash A}\mathsf{C}}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}\mathsf{K}^{\Box} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Box\Gamma,\Diamond B,\Box C,\Box C\Box\vdash\Diamond A}\mathsf{K}^{\Box}}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}\mathsf{C}$
Jumps	

$\equiv_{\mathsf{CP}} := (\equiv \cup \equiv_{\mathsf{c}} \cup \equiv_{\mathsf{e}})$ $\equiv_{\lambda} := (\equiv_{\mathsf{CP}} \cup \equiv_{\mathsf{u}})$ $\equiv_{\mathsf{WIS}} := (\equiv_{\lambda} \cup \equiv_{\Box\mathsf{c}})$

- TABLEAUX 2023 [A. & D. Catta & F. Olimpieri]:
  Canonicity of Proofs in Constructive Modal Logic

Very fast overview of the results

Constructive Modal Logic CK (the $\Box$-fragment)

Formulas/Types: $A, B ::= a \mid A \supset B \mid \Box A$

Terms: $M, N \coloneqq x \mid \lambda x.M \mid (MN) \mid M\left[\vec{N}/\vec{x}\right]_{\blacksquare}$

$$\frac{}{a \vdash a}\,\mathsf{AX} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,\supset^{\mathsf{R}} \quad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\,\supset^{\mathsf{L}} \quad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathsf{C} \quad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathsf{W} \quad \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\mathsf{K}^{\Box}$$

Idea: we need additional reductions (and fix the system accordingly):

Ground Steps:

$$(\lambda x.M)N \rightsquigarrow_{\beta_1} M\{N/x\}$$

$$M\left[\vec{P}, R\left[\vec{N}/\vec{z}\right]_{\blacksquare}, \vec{Q}/\vec{x}, y, \vec{w}\right]_{\blacksquare} \rightsquigarrow_{\beta_2} M\{R/y\}\left[\vec{P}, \vec{N}, \vec{Q}/\vec{x}, \vec{z}, \vec{w}\right]_{\blacksquare}$$

$$M \rightsquigarrow_{\eta_1} \lambda x.Mx \qquad \text{if } \Gamma \vdash M : A \to B,\ x \notin \mathtt{FV}(M) \text{ and } M \notin \Lambda^{\lambda}$$

$$M \rightsquigarrow_{\eta_2} x[M/x]_{\blacksquare} \qquad \text{if } \Gamma \vdash M : \Box A,\ x \notin \mathtt{FV}(M) \text{ and } M \notin \Lambda^{\blacksquare}$$

$$M\left[\vec{P}, N, \vec{Q}/\vec{x}, y, \vec{z}\right]_{\blacksquare} \rightsquigarrow_{\kappa_1} M\left[\vec{P}, \vec{Q}/\vec{x}, \vec{z}\right]_{\blacksquare}$$

$$M\left[\vec{P}, N, N, \vec{Q}/\vec{x}, y_1, y_2, \vec{z}\right]_{\blacksquare} \rightsquigarrow_{\kappa_2} M\{v, v/y_1, y_2\}\left[\vec{P}, N, \vec{Q}/\vec{x}, v, \vec{z}\right]_{\blacksquare}$$

Reduction Steps in Contexts:

$$\frac{M \rightsquigarrow_{\beta_i} N}{\mathbf{C}[M] \rightsquigarrow_{\beta} \mathbf{C}[N]}\ i \in \{1,2\} \qquad \frac{M \rightsquigarrow_{\kappa_i} N}{\mathbf{C}[M] \rightsquigarrow_{\kappa} \mathbf{C}[N]}\ i \in \{1,2\} \qquad \frac{M \rightsquigarrow_{\eta_1} N}{\mathbf{E}[M] \rightsquigarrow_{\eta} \mathbf{E}[N]} \qquad \frac{M \rightsquigarrow_{\eta_2} N}{\mathbf{D}[M] \rightsquigarrow_{\eta} \mathbf{D}[N]}$$

with $\mathbf{C}[\circ] \in$ CwH and $\mathbf{E}[\circ] \in \mathrm{CwH}_{\eta_1}$ and $\mathbf{D}[\circ] \in \mathrm{CwH}_{\eta_2}$

Results:

- The reduction relation $\left(\rightsquigarrow_\beta \cup \rightsquigarrow_\eta \cup \rightsquigarrow_\kappa\right)$ is confluent;
- Inductive definition of $\beta\eta\kappa$-normal forms;
- A focused-like (sound and complete) typing system for $\beta\eta\kappa$-normal terms CK^F;
- A one-to-one correspondence between $\beta\eta\kappa$-normal terms and derivations in CK^F;
- A one-to-one correspondence between derivations in CK^F and CK-WISs;

Theorem

There is a one-to-one correspondence between $\beta\eta\kappa$-normal terms and CK-WIS*s.*

Related works/Works in Progress:

- Combinatorial Proofs and Game Semantics for CS4 (and normalization)
- Extend results on λ-calculus for CK (include $\Diamond$ and $\wedge$)
- Re-study categorical semantics (!)

Thanks

Thanks

Questions?