An Introduction to Combinatorial Proofs

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Why Combinatorial Proofs?

Definition (Proof Theory)

Proof theory is the branch of mathematical logic that studies proofs as formal mathematical objects.

Pythagorean theorem

There are many different proofs of the Pythagorean theorem

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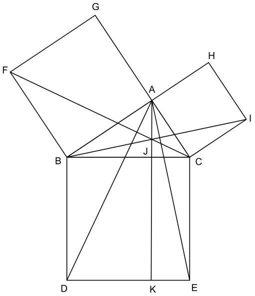

Pythagorean theorem

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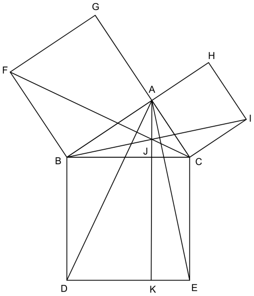

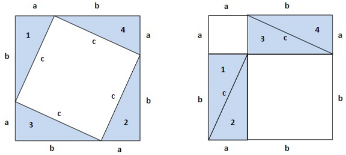

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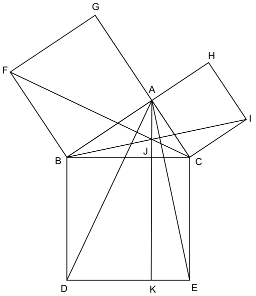

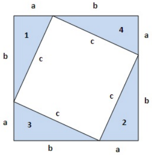

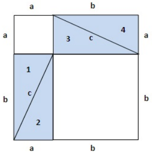

More proofs (122) available at

`http://www.cut-the-knot.org/pythagoras/index.shtml`

Pythagorean theorem

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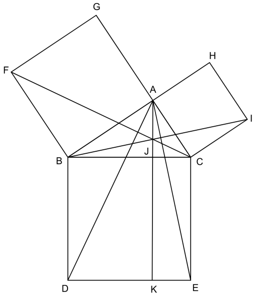

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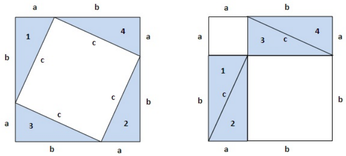

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Proof theory is the branch of mathematical logic that studies proofs as formal mathematical objects.

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PROBLEM

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PROBLEM

We do not have a “nice” representation of the basic object

“[God] caused a tumult among them, by producing in them diverse languages, and causing that, through the multitude of those languages, they should not be able to understand one another.”
(Flavius Josephus, Antiquities of the Jews, c. 94 CE)

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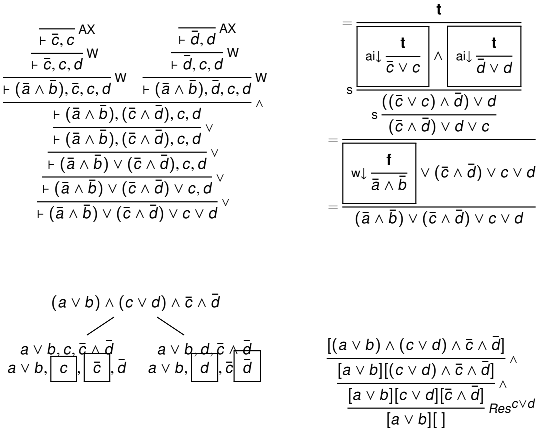

We consider some derivations to be the same proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a,\bar{a}}\,\text{AX} \qquad \dfrac{}{\bar{b},b}\,\text{AX}}{a,\bar{a}\otimes\bar{b},b}\,\otimes}{a \parr (\bar{a}\otimes\bar{b}),b}\,\parr \qquad \dfrac{\dfrac{}{c,\bar{c}}\,\text{AX} \qquad \dfrac{}{\bar{d},d}\,\text{AX}}{c,\bar{c}\otimes\bar{d},d}\,\otimes}{a \parr (\bar{a}\otimes\bar{b}),b\otimes c,d,\bar{c}\otimes\bar{d}}\,\otimes}{a \parr (\bar{a}\otimes\bar{b}),(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\,\parr
$$

Rules permutations

We consider some derivations to be the same proof:

$$
\dfrac{
  \dfrac{
    \dfrac{\dfrac{}{a,\bar a}\,\mathsf{AX} \qquad \dfrac{}{\bar b, b}\,\mathsf{AX}}{a, \bar a \otimes \bar b, b}\,\otimes
    \qquad
    \dfrac{\dfrac{}{c,\bar c}\,\mathsf{AX} \qquad \dfrac{}{\bar d, d}\,\mathsf{AX}}{c, \bar c \otimes \bar d, d}\,\otimes
  }{a, (\bar a \otimes \bar b), b \otimes c, d, \bar c \otimes \bar d}\,\otimes
}{
  \dfrac{a \parr (\bar a \otimes \bar b), b \otimes c, d, \bar c \otimes \bar d}{a \parr (\bar a \otimes \bar b), (b \otimes c) \parr d, \bar c \otimes \bar d}\,\parr
}\,\parr
$$

We consider some derivations to be the same proof:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$a, \bar{a}$}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$\bar{b}, b$}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$c, \bar{c}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\bar{b}, b \otimes c, \bar{c}$}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$\bar{d}, d$}
\RightLabel{$\otimes$}
\BinaryInfC{$\bar{b}, b \otimes c, \bar{c} \otimes \bar{d}, d$}
\RightLabel{$\otimes$}
\BinaryInfC{$a, \bar{a} \otimes \bar{b}, b \otimes c, d, \bar{c} \otimes \bar{d}$}
\RightLabel{$\parr$}
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\RightLabel{$\parr$}
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\RightLabel{$⅋$}
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$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{a, \bar a}\,{\scriptstyle AX} \quad \dfrac{}{\bar b, b}\,{\scriptstyle AX}
    }{a, \bar a \otimes \bar b, b}\,{\scriptstyle \otimes}
  }{a \parr (\bar a \otimes \bar b), b}\,{\scriptstyle \parr}
  \quad
  \dfrac{
    \dfrac{}{c, \bar c}\,{\scriptstyle AX} \quad \dfrac{}{\bar d, d}\,{\scriptstyle AX}
  }{c, \bar c \otimes \bar d, d}\,{\scriptstyle \otimes}
}{
  \dfrac{a \parr (\bar a \otimes \bar b), b \otimes c, d, \bar c \otimes \bar d}{a \parr (\bar a \otimes \bar b), (b \otimes c) \parr d, \bar c \otimes \bar d}\,{\scriptstyle \parr}
}\,{\scriptstyle \otimes}
\quad \simeq \quad
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    \quad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\bar b, b}\,{\scriptstyle AX} \quad \dfrac{}{c, \bar c}\,{\scriptstyle AX}
        }{\bar b, b \otimes c, \bar c}\,{\scriptstyle \otimes}
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      }{\bar b, b \otimes c, \bar c \otimes \bar d, d}\,{\scriptstyle \otimes}
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}{a \parr (\bar a \otimes \bar b), (b \otimes c) \parr d, \bar c \otimes \bar d}\,{\scriptstyle \parr}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{a,\bar a}\,\mathsf{AX} \qquad \dfrac{}{\bar b, b}\,\mathsf{AX}}{a, \bar a \otimes \bar b, b}\,\otimes}{a \parr (\bar a \otimes \bar b), b}\,\parr \qquad \dfrac{\dfrac{}{c,\bar c}\,\mathsf{AX} \qquad \dfrac{}{\bar d, d}\,\mathsf{AX}}{c, \bar c \otimes \bar d, d}\,\otimes}{\dfrac{a \parr (\bar a \otimes \bar b), b \otimes c, d, \bar c \otimes \bar d}{a \parr (\bar a \otimes \bar b), (b \otimes c) \parr d, \bar c \otimes \bar d}\,\parr}\,\otimes
$$

$$\simeq$$

$$
\dfrac{\dfrac{\dfrac{}{a,\bar a}\,\mathsf{AX} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\bar b, b}\,\mathsf{AX} \qquad \dfrac{}{c,\bar c}\,\mathsf{AX}}{\bar b, b \otimes c, \bar c}\,\otimes \qquad \dfrac{}{\bar d, d}\,\mathsf{AX}}{\bar b, b \otimes c, \bar c \otimes \bar d, d}\,\otimes}{(b \otimes c) \parr d, \bar c \otimes \bar d}\,\parr}{a, \bar a \otimes \bar b, (b \otimes c) \parr d, \bar c \otimes \bar d}\,\otimes}{a \parr (\bar a \otimes \bar b), (b \otimes c) \parr d, \bar c \otimes \bar d}\,\parr
$$

[1] Girard 1987

Proof nets[1]

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{a,\bar a}\ \mathsf{AX} \qquad \overline{\bar b, b}\ \mathsf{AX}}{a, \bar a \otimes \bar b, b}\ \otimes}{a \parr (\bar a \otimes \bar b), b}\ \parr \qquad \dfrac{\overline{c,\bar c}\ \mathsf{AX} \qquad \overline{\bar d, d}\ \mathsf{AX}}{c, \bar c \otimes \bar d, d}\ \otimes}{a \parr (\bar a \otimes \bar b), b \otimes c, d, \bar c \otimes \bar d}\ \otimes}{a \parr (\bar a \otimes \bar b), (b \otimes c) \parr d, \bar c \otimes \bar d}\ \parr$$

$$\simeq$$

$$\dfrac{\dfrac{\overline{a,\bar a}\ \mathsf{AX} \qquad \dfrac{\dfrac{\dfrac{\overline{\bar b, b}\ \mathsf{AX} \qquad \overline{c,\bar c}\ \mathsf{AX}}{\bar b, b \otimes c, \bar c}\ \otimes \qquad \overline{\bar d, d}\ \mathsf{AX}}{\bar b, b \otimes c, \bar c \otimes \bar d, d}\ \otimes}{(b \otimes c) \parr d, \bar c \otimes \bar d}\ \parr}{a, \bar a \otimes \bar b, (b \otimes c) \parr d, \bar c \otimes \bar d}\ \otimes}{a \parr (\bar a \otimes \bar b), (b \otimes c) \parr d, \bar c \otimes \bar d}\ \parr$$

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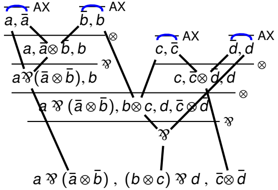
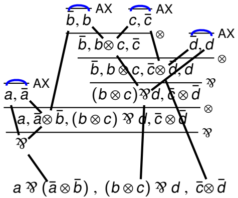
Proof nets[1]



$\simeq$



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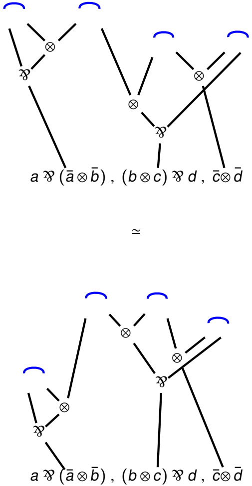

Problem: no proof net* for extensions of MLL (with units or weakening)

[1] Girard 1987

Combinatorial Proofs for Classical Logic

Classical Logic

Formulas

$$A, B := a \mid \bar{a} \mid A \wedge B \mid A \vee B$$

Sequent Calculus LK

$$\text{ax}\,\frac{}{a, \bar{a}} \quad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \quad \wedge\,\frac{\Gamma, A \quad B, \Delta}{\Gamma, A \wedge B, \Delta} \quad \mathsf{W}\,\frac{\Gamma}{\Gamma, A} \quad \mathsf{C}\,\frac{\Gamma, A, A}{\Gamma, A}$$

Theorem

LK *is a sound and complete proof system for classical logic.*

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Theorem

LK *is a sound and complete proof system for classical logic.*

Theorem

Cut elimination holds in LK.

Combinatorial Proofs

Definition

A combinatorial proof of a formula F is an axiom-preserving skew fibration

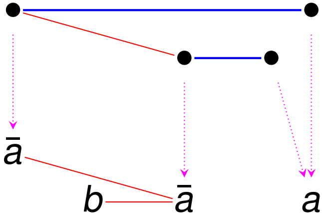
$$f : \mathcal{G} \to [\![F]\!]$$

from a **RB**-cograph $\mathcal{G}$ to the cograph of F.

Ideas:

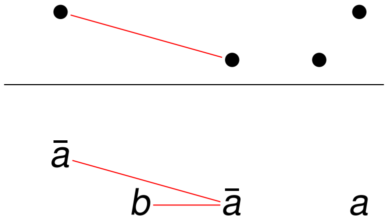
- cograph = graph enconding a formula
- **RB**-cograph = MLL proof nets
- skew fibration = $\{W^{\downarrow}, C^{\downarrow}\}$-derivations (ALL proof nets)

Cographs[2]



[2]Duffin 1965

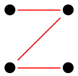
Cographs[2]



[2] Duffin 1965

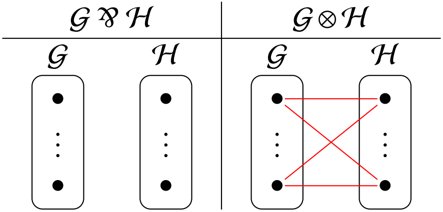
Cographs

Definition

A **cograph** is a graph containing no four vertices such that 

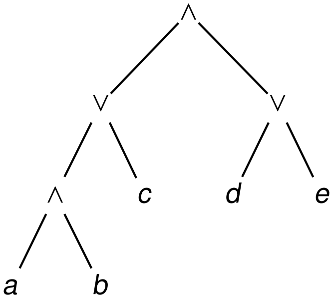
Theorem

A graph is a cograph iff constructed from single-vertices graphs using the graph operations



From formula to cographs

$$((a \wedge b) \vee c) \wedge (d \vee e)$$



a	b	
a	c	
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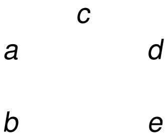
c	d	
c	e	
d	e	



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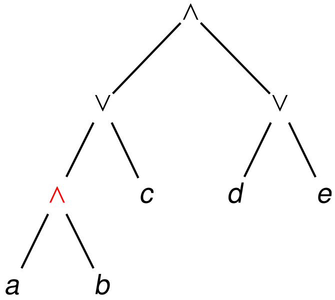

a	b	$\frown$
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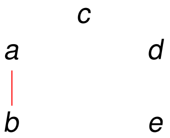

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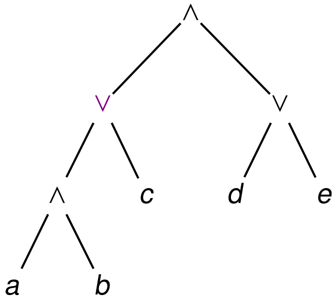

a	b	
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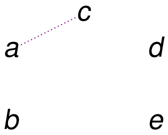

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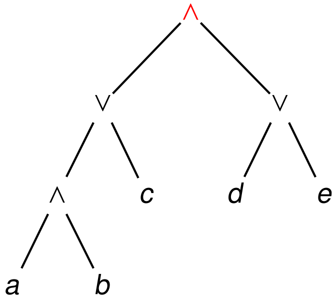

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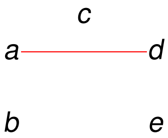

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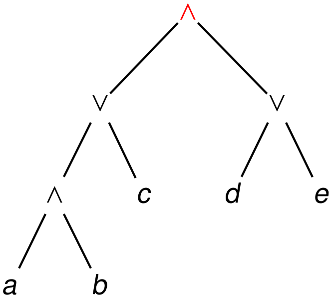

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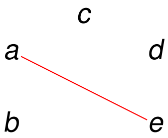

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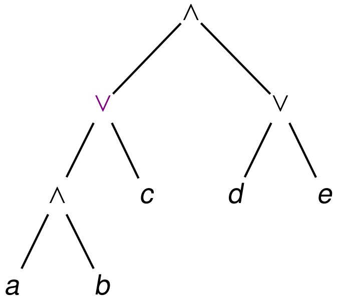

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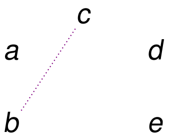

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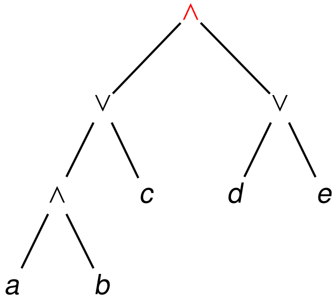

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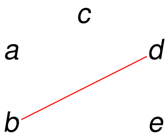

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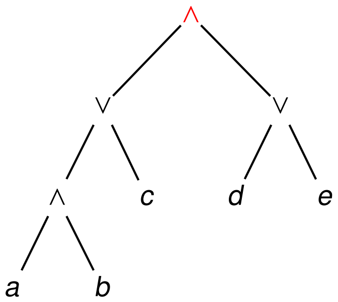

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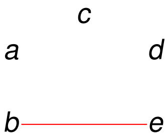

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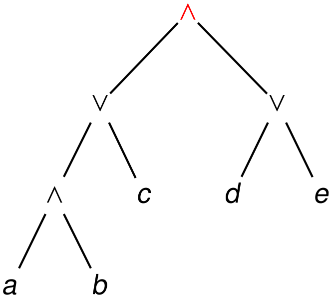

a	b	
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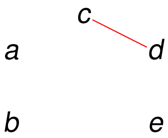

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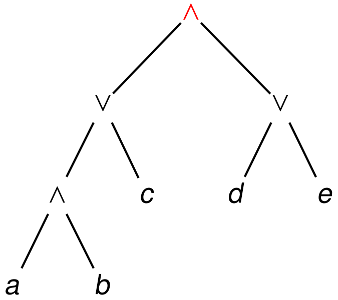

a	b	
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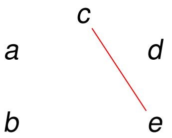
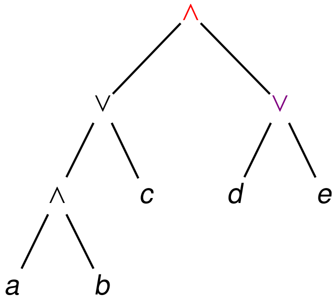
a	d	
a	e	
b	c	
b	d	
b	e	
c	d	
c	e	$\frown$
d	e	

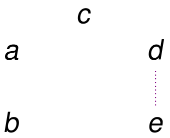


From formula to cographs

$$((a \wedge b) \vee c) \wedge (d \vee e)$$



a	b	
a	c	
a	d	
a	e	
b	c	
b	d	
b	e	
c	d	
c	e	
d	e	$\not\frown$



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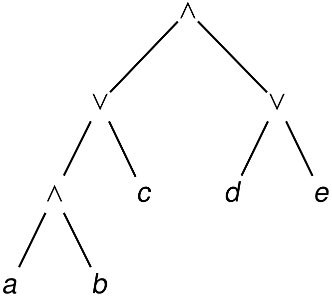

a	b	$\frown$
a	c	$\not\frown$
a	d	$\frown$
a	e	$\frown$
b	c	$\not\frown$
b	d	$\frown$
b	e	$\frown$
c	d	$\frown$
c	e	$\frown$
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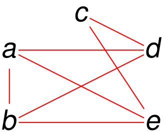
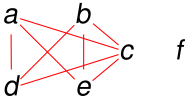
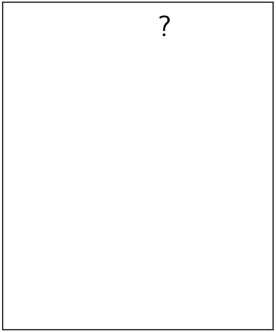


From cographs to formulas

Lemma

If $\mathcal{G}$ is a cograph, then either $\mathcal{G}$ or $\bar{\mathcal{G}}$ is disconnected.





Formula = ?

From cographs to formulas

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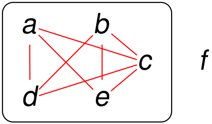

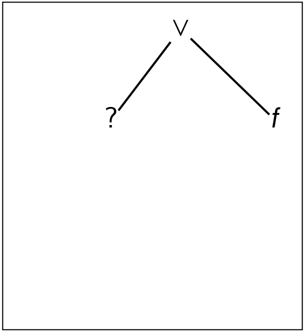

Formula = ? $\vee$ *f*

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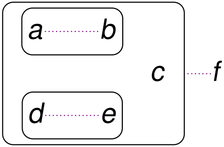

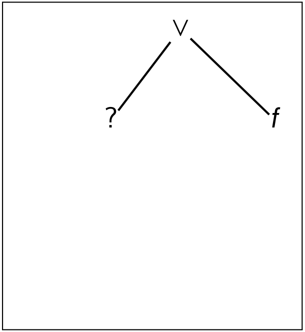

Formula = $? \vee f$

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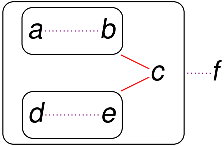

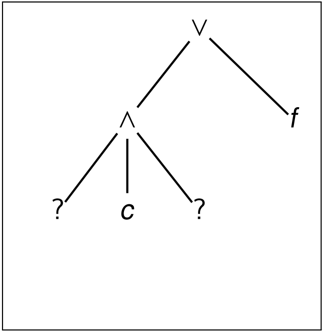

Formula = $(? \wedge c \wedge ?) \vee f$

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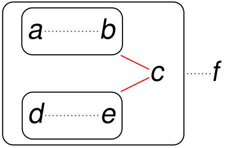
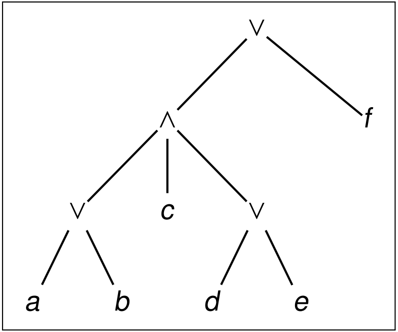




Formula = $((a \vee b) \wedge c \wedge (d \vee e)) \vee f$

Cograph and Formula Isomophism

Definition

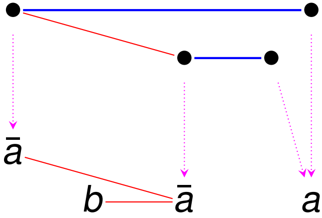
The formula isomorphism $\simeq$ is the equivalence relation generated by:

$$A \wedge B \simeq B \wedge A \qquad A \vee B \simeq B \vee A$$
$$(A \wedge B) \wedge C \simeq A \wedge (B \wedge C) \qquad (A \vee B) \vee C \simeq A \vee (B \vee C)$$

Theorem

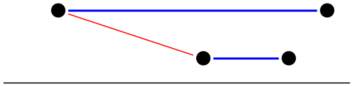
$$F \simeq F' \iff [\![F]\!] = [\![F']\!]$$

RB-cographs[3]



[3] Retoré 1993

RB-cographs[3]



[3] Retoré 1993

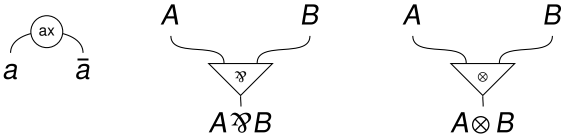
MLL Proof nets

The sequent calculus for LK

$$\text{ax}\,\frac{}{a,\bar{a}} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \wedge\,\frac{\Gamma, A \quad B, \Delta}{\Gamma, A \wedge B, \Delta} \qquad \frac{\Gamma}{\Gamma, A}\,\text{w} \qquad \frac{\Gamma, A, A}{\Gamma, A}\,\text{c}$$

Definition

A **proof structure** is a graph constructed using the following links



A **proof net** is a proof structure encoding a derivation in MLL

MLL Proof nets

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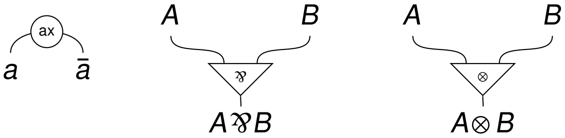

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$$\mathsf{ax}\ \frac{}{a, \bar{a}} \qquad \parr\ \frac{\Gamma, A, B}{\Gamma, A \parr B} \qquad \otimes\ \frac{\Gamma, A \quad B, \Delta}{\Gamma, A \otimes B, \Delta}$$

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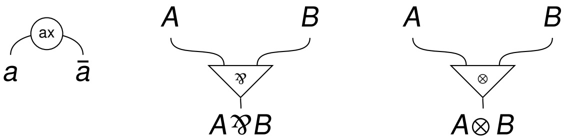

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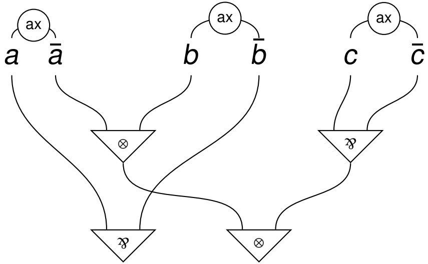
Definition

A proof structure is correct if “pruning” one input from each ⅋-gate we obtain a connected and acyclic graph.



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A proof net is correct iff it is connected and acyclic (for each **switching**).

MLL Proof nets

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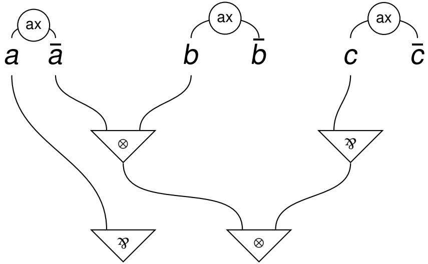

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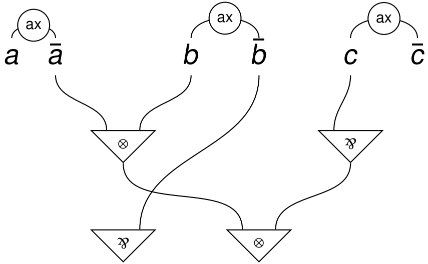

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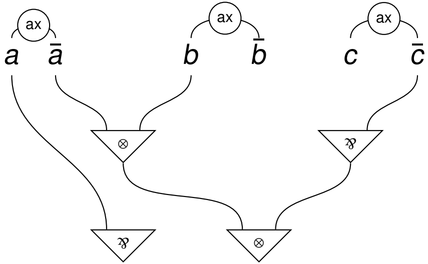

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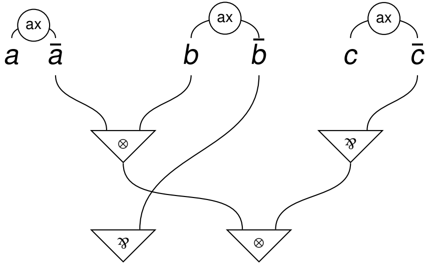

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Handsome proof nets

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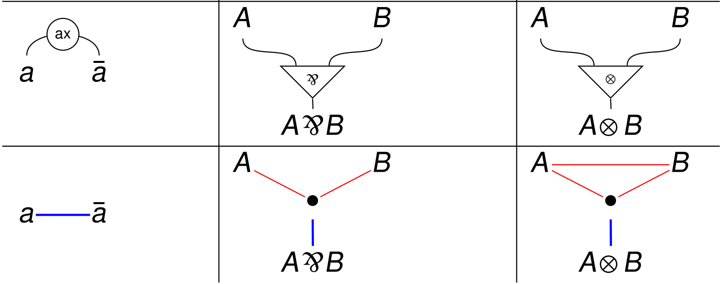

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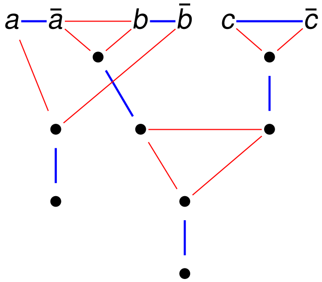
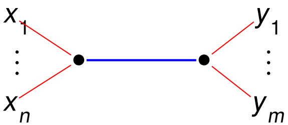


Definition

A **RB**-proof net is correct iff it is æ-connected and æ-acyclic.

Handsome proof nets: unfolding

Unfolding = remove •-vertices from the graph



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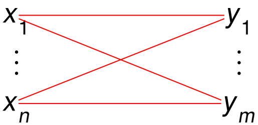

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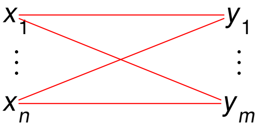

Note: by removing •-vertices we remove all non-axiom ⋎-edges

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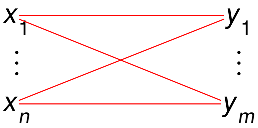
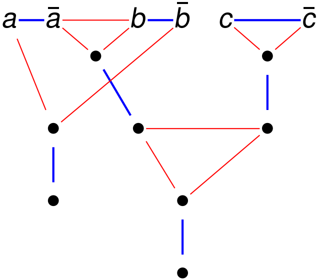


Note: by removing •-vertices we remove all non-axiom $\curlyvee$-edges Note: by removing $\curlyvee$-edges we may introduce bow-ties (see above)

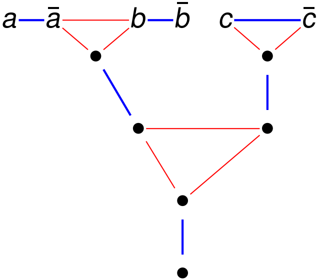
Handsome proof nets: unfolding



Definition

A **RB**-cograph is correct iff it is æ-connected and æ-acyclic .

Handsome proof nets: unfolding



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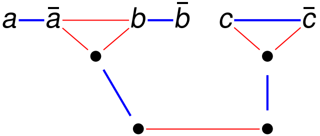

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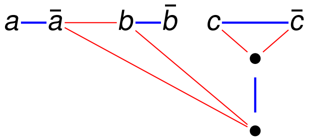

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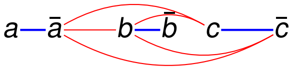

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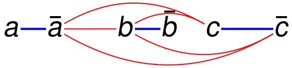

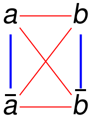

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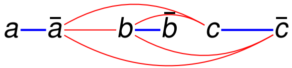
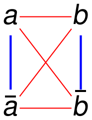
RB-cograph





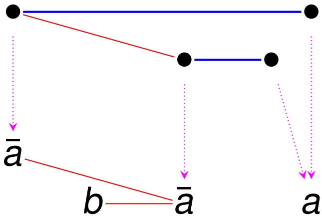
Definition

A **RB**-cograph is correct iff it is æ-connected and æ-acyclic w.r.t. cordless paths.

Theorem

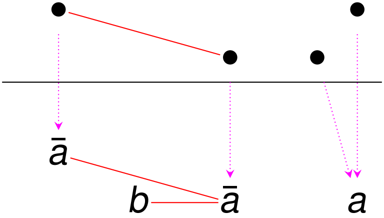
$\overset{\text{MLL}}{\vdash} F \iff$ *exists a correct* ***RB****-cograph* $\langle V, \frown, \curlyvee \rangle$ *s.t.* $[\![F]\!] = \langle V, \frown \rangle$

Skew Fibrations[4]



[4] Hughes 2005; Straßburger RTA2007

Skew Fibrations[4]



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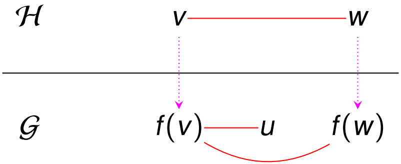
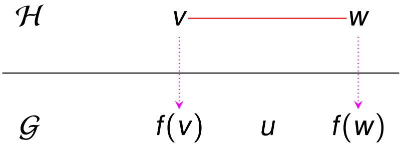


Definition

- A graph **homomorphism** $f\colon \mathcal{H} \to \mathcal{G}$ between two graphs is a map $f\colon V_{\mathcal{H}} \to V_{\mathcal{G}}$ preserving $\frown$-edges;

Skew Fibration



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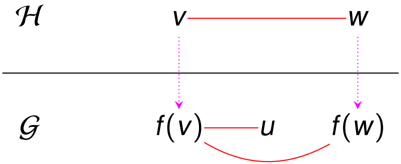

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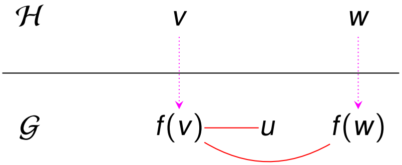

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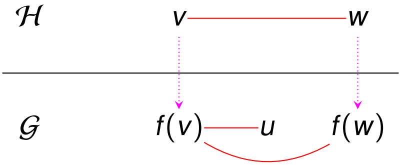

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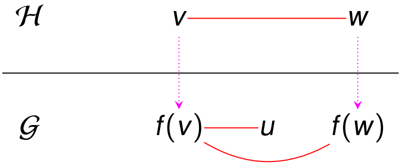

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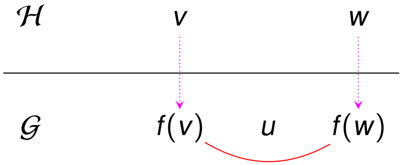
Skew Fibration



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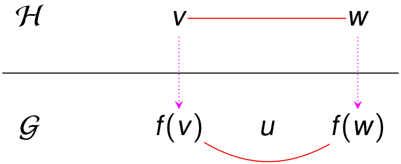
Skew Fibration



Definition

- A graph **homomorphism** $f\colon \mathcal{H} \to \mathcal{G}$ between two graphs is a map $f\colon V_{\mathcal{H}} \to V_{\mathcal{G}}$ preserving $\frown$-edges;
- A **fibration** is an homomorphism $f\colon \mathcal{H} \to \mathcal{G}$ such that

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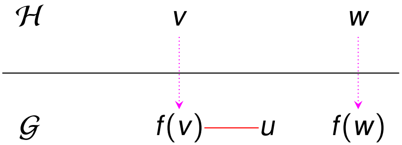

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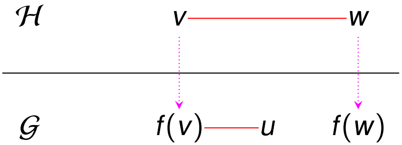

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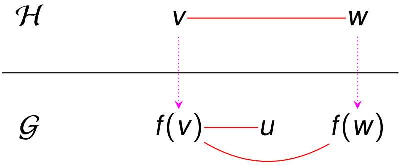

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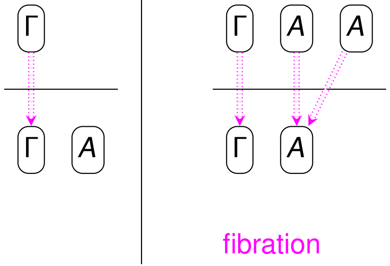
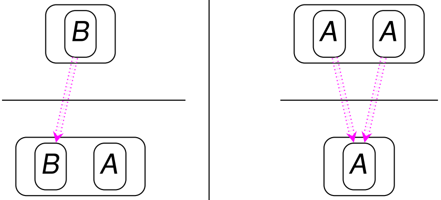
$$\mathsf{w}\,\frac{\Gamma}{\Gamma, A} \qquad\qquad \mathsf{c}\,\frac{\Gamma, A, A}{\Gamma, A}$$



$$\mathsf{w}{\downarrow}\,\frac{\mathcal{C}\{B\}}{\mathcal{C}\{B \vee A\}} \qquad \mathsf{c}{\downarrow}\,\frac{\mathcal{C}\{A \vee A\}}{\mathcal{C}\{A\}}$$



fibration

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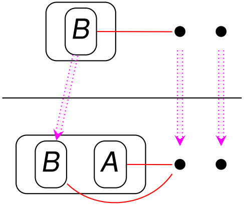

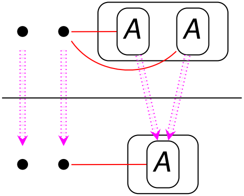

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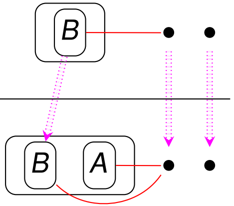
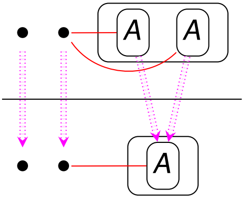
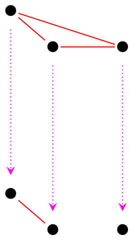


skew



fibration

Skew Fibrations (midterm exam)



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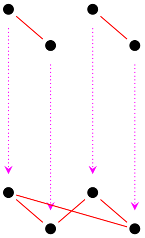
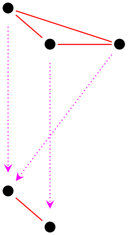
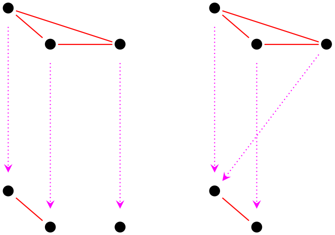


Skew Fibrations (midterm exam)



Skew Fibrations (midterm exam)



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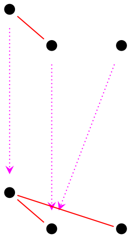

Skew Fibrations (midterm exam)

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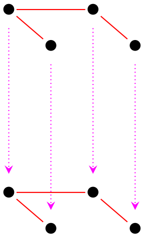
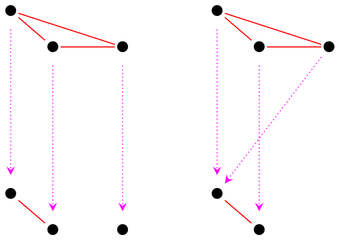
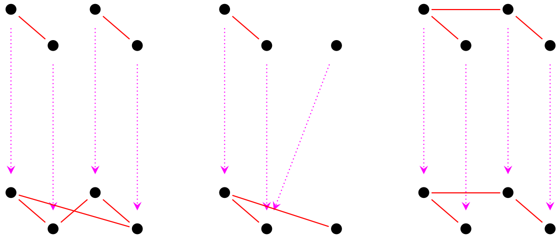


Skew Fibrations (midterm exam)

Is a not skew fibration



Is a skew fibration



Skew Fibrations[5]

Theorem (Decomposition)

$$F' \overset{\{\mathsf{w}^{\downarrow},\mathsf{c}^{\downarrow}\}}{\longmapsto} F \Longrightarrow \textit{there is a skew fibration } f : [\![F']\!] \to [\![F]\!]$$

$$\begin{array}{c} F' \\ \| \{\mathsf{w}^{\downarrow},\mathsf{c}^{\downarrow}\} \\ F \end{array} \Longrightarrow \begin{array}{c} F' \\ \| \mathsf{m}^{\downarrow} \\ G' \\ \| \mathsf{ac}^{\downarrow} \\ G \\ \| \mathsf{w}^{\downarrow} \\ F \end{array} \Longrightarrow \begin{array}{c} [\![F']\!] \\ \downarrow \text{bijective} \\ [\![G']\!] \\ \downarrow \text{surjective} \\ [\![G]\!] \\ \downarrow \text{injecive} \\ [\![F]\!] \end{array}$$

$$\mathsf{m}^{\downarrow}\frac{\mathcal{C}\{(A \wedge C) \vee (B \wedge D)\}}{\mathcal{C}\{(A \vee B) \wedge (C \vee D)\}} \qquad \mathsf{ac}^{\downarrow}\frac{\mathcal{C}\{a \vee a\}}{\mathcal{C}\{a\}} \qquad \mathsf{w}^{\downarrow}\frac{\mathcal{C}\{B\}}{\mathcal{C}\{A \vee B\}}$$

[5] Hughes 2005 ; Straßburger RTA2007

Reassembling the pieces

Combinatorial Proofs

What we have:

- **RB**-cograph: a graphical syntax for MLL proofs
- Skew fibrations: graph homomorphisms representing $\{\mathsf{W}^{\downarrow}, \mathsf{C}^{\downarrow}\}$-derivations

What do we what:

- Combine them to have a graphical syntax for $\mathsf{LK} = \mathsf{MLL} \cup \{\mathsf{W}, \mathsf{C}\}$

$$\wedge\frac{\mathsf{C}\dfrac{\overset{\displaystyle\Pi}{\Gamma, A, A}}{\Gamma, A} \qquad \vee\dfrac{\mathsf{W}\dfrac{\overset{\displaystyle\Pi}{\Delta, B}}{\Delta, B, C}}{\Delta, B \vee C}}{\Gamma, \Delta, A \wedge (B \vee C)} \quad \rightsquigarrow ?$$

What we have:

- **RB**-cograph: a graphical syntax for MLL proofs
- Skew fibrations: graph homomorphisms representing $\{W^{\downarrow}, C^{\downarrow}\}$-derivations

What do we what:

- Combine them to have a graphical syntax for $\mathsf{LK} = \mathsf{MLL} \cup \{\mathsf{W}, \mathsf{C}\}$

$$
\wedge\dfrac{\mathsf{C}\dfrac{\overset{\Vert}{\Gamma, A, A}}{\Gamma, A} \qquad \vee\dfrac{\mathsf{W}\dfrac{\overset{\Vert}{\Delta, B}}{\Delta, B, C}}{\Delta, B \vee C}}{\Gamma, \Delta, A \wedge (B \vee C)}
\quad \rightsquigarrow \quad
\mathsf{C}^{\downarrow}\dfrac{\mathsf{W}^{\downarrow}\dfrac{\wedge\dfrac{\vee\dfrac{\overset{\Vert}{\Gamma, A, A}}{\Gamma, A \vee A} \qquad \overset{\Vert}{\Delta, B}}{\Gamma, \Delta, (A \vee A) \wedge B}}{\Gamma, \Delta, (A \vee A) \wedge (B \vee C)}}{\Gamma, \Delta, A \wedge (B \vee C)}
$$

Combinatorial Proofs

Theorem (Decomposition)

$$\overset{\mathsf{LK}}{\vdash} F \Longrightarrow \overset{\mathsf{MLL}}{\vdash} F', \overset{\{\mathsf{W}^{\downarrow},\mathsf{C}^{\downarrow}\}}{\vdash} F$$

$$\begin{array}{c}\Vert \mathsf{LK} \\ F\end{array}$$

Theorem

Every LK *derivation can be represented by a combinatorial proof*

Combinatorial Proofs

Theorem (Decomposition)

$$\vdash^{\mathsf{LK}} F \Longrightarrow \vdash^{\mathsf{MLL}} F' \vdash^{\{\mathsf{W}^{\downarrow},\mathsf{C}^{\downarrow}\}} F$$

$$\begin{array}{c} \Vert\,\mathsf{LK} \\ F \end{array} \quad \Longrightarrow \quad \begin{array}{c} \mathcal{D}' \Vert\,\mathsf{MLL} \\ F' \\ \mathcal{D} \Vert\,\{\mathsf{w}^{\downarrow},\mathsf{c}^{\downarrow}\} \\ F \end{array}$$

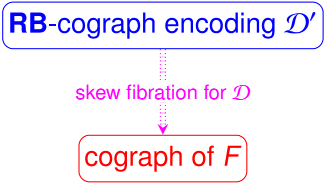
Theorem

Every LK *derivation can be represented by a combinatorial proof*

Theorem (Decomposition)

$$\overset{\mathsf{LK}}{\vdash} F \Longrightarrow \overset{\mathsf{MLL}}{\vdash} F' \overset{\{\mathsf{W}^{\downarrow},\mathsf{C}^{\downarrow}\}}{\vdash} F$$

$$\begin{array}{c} \Big\Vert\,\mathsf{LK} \\ F \end{array} \quad\Longrightarrow\quad \begin{array}{c} \mathcal{D}'\Big\Vert\,\mathsf{MLL} \\ F' \\ \mathcal{D}\Big\Vert\,\{\mathsf{W}^{\downarrow},\mathsf{C}^{\downarrow}\} \\ F \end{array} \quad\Longrightarrow$$



Theorem

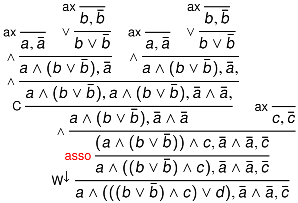
Every LK *derivation can be represented by a combinatorial proof*

Combinatorial Proofs

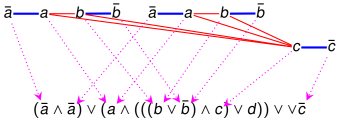
Theorem

Every combinatorial proof can be sequentialized into a derivation in LK ∪ {cut}

Where is the problem?
Hughes's example:



$\Longleftrightarrow$



Theorem

$$F' \overset{\{\mathsf{W}^{\downarrow},\mathsf{C}^{\downarrow},\equiv\}}{\longmapsto} F \iff \textit{there is a skew fibration } {\color{magenta}f} : [\![F']\!] \to [\![F]\!]$$

Combinatorial Proofs form a Proof System

Fact (Cook-Reckhow)

Check whether a syntactic object represents a valid proof can be done by means of a polynomial time algorithm.

- Check if a graph is a cograph
- Check if a **RB**-cograph is æ-connected and æ-acyclic
- Check if a map $f : \mathcal{H} \to \mathcal{G}$ between cograph is a skew fibration
- Check if f is axiom-preserving

Theorem

Combinatorial Proofs form a proof system for classical logic.

The (current) realm of Combinatorial Proofs

CPs for Relevant and Affine Logics[6]

- Relevant Logic = LK without weakening
- Affine Logic = LK without contraction

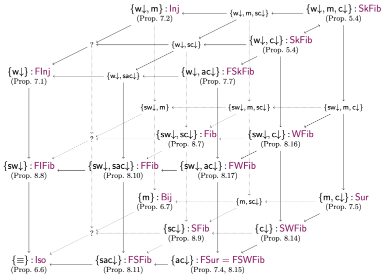


*figure from Ralph and Straßburger paper

- Entailment Logic (non associative connectives)

[6] Ralph & Straßburger Tablueaux2019; Acclavio & Straßburger Wollic2019

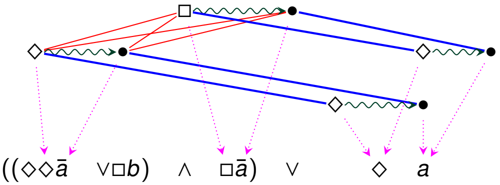
Modal Logic S4[7]

Modal Formulas

$$A, B := a \mid \bar{a} \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A$$

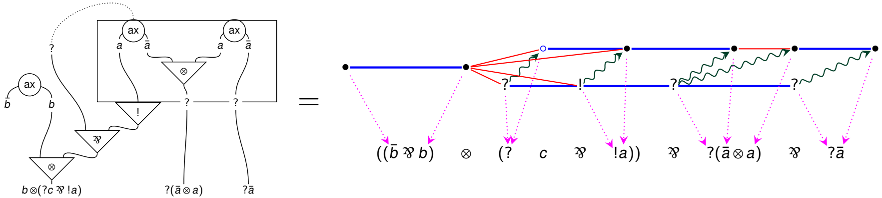
Sequent Calculus Rules

$$\mathsf{LK} \cup \left\{ \mathsf{K}\,\frac{A, \Gamma}{\Box A, \Diamond \Gamma} \quad , \quad \mathsf{D}\,\frac{A, \Gamma}{\Diamond A, \Diamond \Gamma} \quad , \quad \mathsf{T}^{\downarrow}\,\frac{\mathcal{C}\{A\}}{\mathcal{C}\{\Diamond A\}} \quad , \quad 4^{\downarrow}\,\frac{\mathcal{C}\{\Diamond\Diamond A\}}{\mathcal{C}\{\Diamond A\}} \right\}$$



[7] Acclavio & Straßburger Tabuleaux2019

Multiplicative Linear Logic with Exponentials[8]



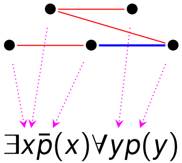
[8] Acclavio TLLA2020

First Order Classical Logic [9]

Formulas

$$
\begin{aligned}
t &:= c \mid f(t_1, \ldots, t_n) \\
a &:= p(t_1, \ldots, t_n) \mid \bar{p}(t_1, \ldots, t_n) \\
A, B &:= a \mid A \wedge B \mid A \vee B \mid \forall x A \mid \exists x A
\end{aligned}
$$

Rules $\mathsf{LK} \cup \left\{ \quad \exists \frac{\Gamma, A[x/t]}{\Gamma, \exists x.A} \quad , \quad \forall \frac{\Gamma, A}{\Gamma, \forall x.A} \; x \text{ not free in } \Gamma \quad \right\}$



[9] Hughes 2019; Hughes & Straßburger & Wu LICS2021

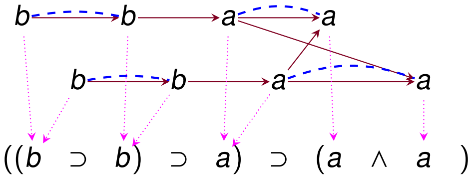
Formulas

$$A, B \coloneqq a \mid A \wedge B \mid A \supset B$$

Sequent Calculus Rules

$$\frac{}{a \vdash a}\,\mathsf{ax} \qquad \frac{\Gamma, B \vdash A}{\Gamma \vdash B \supset A}\supset^{\mathsf{R}} \qquad \frac{\Gamma, B, C \vdash A}{\Gamma, B \wedge C \vdash A}\wedge^{\mathsf{L}} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\wedge^{\mathsf{R}} \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\supset^{\mathsf{L}}$$

$$\frac{}{\vdash 1}\,1 \qquad \frac{\Gamma, B, B \vdash A}{\Gamma, B \vdash A}\,\mathsf{C} \qquad \frac{\Gamma \vdash A}{\Gamma, B \vdash A}\,\mathsf{W}$$



[10] Heijltjes, Hughes & Straßburger LICS2019

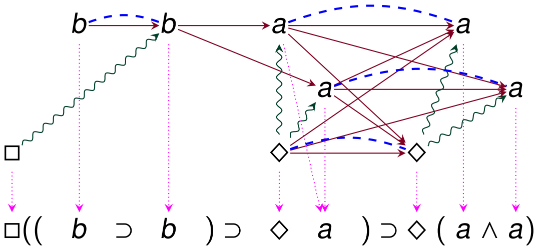
Constructive Modal Logic[11]

Modal Formulas

$$A, B \coloneqq a \mid A \wedge B \mid A \supset B \mid \Box A \mid \Diamond A \mid 1$$

Additional Sequent Calculus Rules

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\mathsf{K}_{\Box} \qquad \frac{B, \Gamma \vdash A}{\Diamond B, \Box\Gamma \vdash \Diamond A}\,\mathsf{K}_{\Diamond} \qquad \frac{B, \Gamma \vdash A}{\Box\Gamma \vdash \Diamond A}\,\mathsf{D}$$



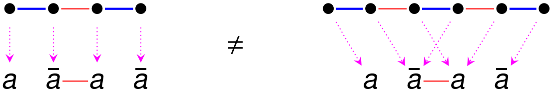
[11] Acclavio, Catta & Straßburger 2021

Combinatorial Proofs and Proof Equivalence

Combinatorial Proofs and Proof equivalence

Claim

Two proofs are the same iff they can be represented by the same CP



- Combinatorial Proofs and sequent calculus[12]
- Combinatorial Proofs and deep inference[13]
- Combinatorial Proofs and Resolution and Analytic Tableaux[14]

[12] Hughes, 2005

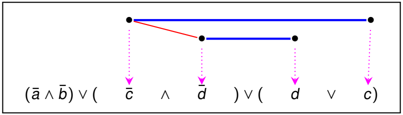
[13] Straßburger, FSCD2017

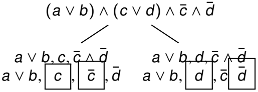
[14] Acclavio & Straßburger, IJCAR2018

Comparing Proofs from Different Proof Systems

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bar c, c}\,\mathsf{AX}}{\vdash \bar c, c, d}\,\mathsf{W}}{\vdash (\bar a\wedge\bar b),\bar c,c,d}\,\mathsf{W} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \bar d, d}\,\mathsf{AX}}{\vdash \bar d, c, d}\,\mathsf{W}}{\vdash (\bar a\wedge\bar b),\bar d,c,d}\,\mathsf{W}}{\vdash (\bar a\wedge\bar b),(\bar c\wedge\bar d),c,d}\,\wedge}{\vdash (\bar a\wedge\bar b),(\bar c\wedge\bar d),c,d}\,\vee}{\vdash (\bar a\wedge\bar b)\vee(\bar c\wedge\bar d),c,d}\,\vee}{\vdash (\bar a\wedge\bar b)\vee(\bar c\wedge\bar d)\vee c,d}\,\vee}{\vdash (\bar a\wedge\bar b)\vee(\bar c\wedge\bar d)\vee c\vee d}\,\vee$$

$$\mathbf{t} \;=\; \left[\mathsf{ai}{\downarrow}\dfrac{\mathbf{t}}{\bar c\vee c}\right]\wedge\left[\mathsf{ai}{\downarrow}\dfrac{\mathbf{t}}{\bar d\vee d}\right] \;\overset{\mathsf{s}}{\longrightarrow}\; ((\bar c\vee c)\wedge\bar d)\vee d \;\overset{\mathsf{s}}{\longrightarrow}\; (\bar c\wedge\bar d)\vee d\vee c \;=\; \left[\mathsf{w}{\downarrow}\dfrac{\mathbf{f}}{\bar a\wedge\bar b}\right]\vee(\bar c\wedge\bar d)\vee c\vee d \;=\; (\bar a\wedge\bar b)\vee(\bar c\wedge\bar d)\vee c\vee d$$





$$\dfrac{\dfrac{\dfrac{[(a\vee b)\wedge(c\vee d)\wedge\bar c\wedge\bar d]}{[a\vee b][(c\vee d)\wedge\bar c\wedge\bar d]}\,\wedge}{[a\vee b][c\vee d][\bar c\wedge\bar d]}\,\wedge}{[a\vee b][\,]}\,Res^{c\vee d}$$

Proof Equivalence in Sequent Calculus

Rules permutations

$$\wedge\frac{\Delta, A \qquad \wedge\dfrac{\Delta, B, C \quad \Sigma, D}{\Delta, \Sigma, B, C \wedge D}}{\Gamma, \Delta, \Sigma, A \wedge B, C \wedge D} \simeq \wedge\frac{\wedge\dfrac{\Delta, A \quad \Sigma, B, C}{\Gamma, \Delta, A \wedge B, C} \qquad \Sigma, D}{\Gamma, \Delta, \Sigma, A \wedge B, C \wedge D}$$

$$\tau\frac{\rho\dfrac{\Gamma, \Delta, \Sigma}{\Gamma, A, \Sigma}}{\Gamma, A, B} \simeq \rho\frac{\tau\dfrac{\Gamma, \Delta, \Sigma}{\Gamma, \Delta, B}}{\Gamma, A, B}$$

$$\wedge\frac{\rho\dfrac{\Gamma, \Delta, B}{\Gamma, A, B} \qquad \Delta, C}{\Gamma, \Delta, A, B \wedge C} \simeq \rho\frac{\wedge\dfrac{\Gamma, \Delta, B \quad \Delta, C}{\Gamma, \Delta, \Delta, B \wedge C}}{\Gamma, \Delta, A, B \wedge C}$$

Proof Equivalence in Sequent Calculus

Rules permutations

$$\wedge\dfrac{\Delta, A \qquad \wedge\dfrac{\Delta, B, C \quad \Sigma, D}{\Delta, \Sigma, B, C \wedge D}}{\Gamma, \Delta, \Sigma, A \wedge B, C \wedge D} \simeq \wedge\dfrac{\wedge\dfrac{\Delta, A \quad \Sigma, B, C}{\Gamma, \Delta, A \wedge B, C} \quad \Sigma, D}{\Gamma, \Delta, \Sigma, A \wedge B, C \wedge D} \qquad \tau\dfrac{\rho\dfrac{\Gamma, \Delta, \Sigma}{\Gamma, A, \Sigma}}{\Gamma, A, B} \simeq \rho\dfrac{\tau\dfrac{\Gamma, \Delta, \Sigma}{\Gamma, \Delta, B}}{\Gamma, A, B} \qquad \wedge\dfrac{\rho\dfrac{\Gamma, \Delta, B}{\Gamma, A, B} \quad \Delta, C}{\Gamma, \Delta, A, B \wedge C} \simeq \rho\dfrac{\wedge\dfrac{\Gamma, \Delta, B \quad \Delta, C}{\Gamma, \Delta, \Delta, B \wedge C}}{\Gamma, \Delta, A, B \wedge C}$$

Comonoid transformations

$$\dfrac{\dfrac{\Gamma, A_1, A_2, A_3}{\Gamma, A_1, A}\,\mathsf{C}}{\Gamma, A}\,\mathsf{C} \simeq \dfrac{\dfrac{\Gamma, A_1, A_2, A_3}{\Gamma, A, A_3}\,\mathsf{C}}{\Gamma, A}\,\mathsf{C} \qquad \dfrac{\dfrac{\Gamma, A, A}{\Gamma, A}\,\mathsf{C}}{\Gamma, A, A}\,\mathsf{W} \simeq \Gamma, A, A \qquad \dfrac{\dfrac{\Gamma, A}{\Gamma, A, A}\,\mathsf{W}}{\Gamma, A}\,\mathsf{C} \simeq \Gamma, A$$

Proof Equivalence in Sequent Calculus

Rules permutations

$$\wedge\dfrac{\Delta, A \qquad \wedge\dfrac{\Delta, B, C \quad \Sigma, D}{\Delta, \Sigma, B, C \wedge D}}{\Gamma, \Delta, \Sigma, A \wedge B, C \wedge D} \simeq \wedge\dfrac{\wedge\dfrac{\Delta, A \quad \Sigma, B, C}{\Gamma, \Delta, A \wedge B, C} \quad \Sigma, D}{\Gamma, \Delta, \Sigma, A \wedge B, C \wedge D}$$

$$\tau\dfrac{\rho\dfrac{\Gamma, \Delta, \Sigma}{\Gamma, A, \Sigma}}{\Gamma, A, B} \simeq \rho\dfrac{\tau\dfrac{\Gamma, \Delta, \Sigma}{\Gamma, \Delta, B}}{\Gamma, A, B}$$

$$\wedge\dfrac{\rho\dfrac{\Gamma, \Delta, B}{\Gamma, A, B} \quad \Delta, C}{\Gamma, \Delta, A, B \wedge C} \simeq \rho\dfrac{\wedge\dfrac{\Gamma, \Delta, B \quad \Delta, C}{\Gamma, \Delta, \Delta, B \wedge C}}{\Gamma, \Delta, A, B \wedge C}$$

Comonoid transformations

$$\dfrac{\dfrac{\Gamma, A_1, A_2, A_3}{\Gamma, A_1, A}\,\mathsf{c}}{\Gamma, A}\,\mathsf{c} \simeq \dfrac{\dfrac{\Gamma, A_1, A_2, A_3}{\Gamma, A, A_3}\,\mathsf{c}}{\Gamma, A}\,\mathsf{c}$$

$$\dfrac{\dfrac{\Gamma, A, A}{\Gamma, A}\,\mathsf{c}}{\Gamma, A, A}\,\mathsf{w} \simeq \Gamma, A, A$$

$$\dfrac{\dfrac{\Gamma, A}{\Gamma, A, A}\,\mathsf{w}}{\Gamma, A}\,\mathsf{c} \simeq \Gamma, A$$

$$\wedge\dfrac{\overset{\pi}{\Vert}\;\Gamma, A \qquad \mathsf{c}\dfrac{\overset{\pi'}{\Vert}\;B, B, \Delta}{B, \Delta}}{\Gamma, A \wedge B, \Delta} \simeq \mathsf{c}\dfrac{\wedge\dfrac{\overset{\pi}{\Vert}\;\Gamma, A \qquad \wedge\dfrac{\overset{\pi}{\Vert}\;\Gamma, A \quad \overset{\pi'}{\Vert}\;B, B, \Delta}{\Gamma, A \wedge B, B, \Delta}}{\Gamma, A \wedge B, \Gamma, A \wedge B, \Delta}}{\Gamma, A \wedge B, \Delta}$$

unfolding

$$\wedge\dfrac{\overset{\pi}{\Vert}\;\Gamma, A \qquad \mathsf{w}\dfrac{\overset{\pi'}{\Vert}\;\Delta}{B, \Delta}}{\Gamma, A \wedge B, \Delta} \simeq \mathsf{w}\dfrac{\overset{\pi'}{\Vert}\;\Delta}{\Gamma, A \wedge B, \Delta}$$

excising

Proof Equivalence in LJ

Definition

The proof equivalence in

Natural Deduction = λ-calculus = Winning Innocent Strategies

is given by

Rules permutations + Comonoid transformations + Unfolding + Excising

Definition

The proof equivalence in

Intuitionistic Combinatorial Proofs

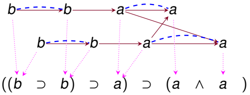
is given by

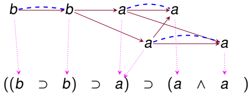
Rules permutations + Comonoid transformations + Exchising

Proof Equivalence in LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}{\scriptstyle \mathsf{AX}}}{\vdash b \supset b}{\scriptstyle \supset^{R}} \quad \dfrac{}{a \vdash a}{\scriptstyle \mathsf{AX}}}{(b \supset b) \supset a \vdash a}{\scriptstyle \supset^{L}} \quad \dfrac{\dfrac{\dfrac{}{b \vdash b}{\scriptstyle \mathsf{AX}}}{\vdash b \supset b}{\scriptstyle \supset^{R}} \quad \dfrac{}{a \vdash a}{\scriptstyle \mathsf{AX}}}{(b \supset b) \supset a \vdash a}{\scriptstyle \supset^{L}}}{(b \supset b) \supset a, (b \supset b) \supset a \vdash a \wedge a}{\scriptstyle \wedge^{R}}}{(b \supset b) \supset a \vdash a \wedge a}{\scriptstyle \mathsf{C}}$$

$$\dfrac{\dfrac{\dfrac{}{b \vdash b}{\scriptstyle \mathsf{AX}}}{\vdash b \supset b}{\scriptstyle \supset^{R}} \quad \dfrac{\dfrac{\dfrac{}{a \vdash a}{\scriptstyle \mathsf{AX}} \quad \dfrac{}{a \vdash a}{\scriptstyle \mathsf{AX}}}{a, a \vdash a \wedge a}{\scriptstyle \wedge^{L}}}{a \vdash a \wedge a}{\scriptstyle \mathsf{C}}}{(b \supset b) \supset a \vdash a \wedge a}{\scriptstyle \supset^{L}} \qquad \dfrac{\dfrac{\dfrac{}{b \vdash b}{\scriptstyle \mathsf{AX}}}{\vdash b \supset b}{\scriptstyle \supset^{R}} \quad \dfrac{\dfrac{\dfrac{}{a \vdash a}{\scriptstyle \mathsf{AX}} \quad \dfrac{}{a \vdash a}{\scriptstyle \mathsf{AX}}}{a, a \vdash a \wedge a}{\scriptstyle \wedge^{L}}}{a \vdash a \wedge a}{\scriptstyle \mathsf{C}}}{(b \supset b) \supset a \vdash a \wedge a}{\scriptstyle \supset^{L}}$$





Both these proofs correspond to a derivation of
$$f\colon (b \supset b) \supset a \vdash \Big(f(\lambda x.x), f(\lambda y.y)\Big)\colon a \wedge a$$

Are two proofs using different amounts of the same resources equal?

Compositionality

How to represent cut[15]

Combinatorial proofs allows to represent cut-free proofs

$\rightsquigarrow$ $\rightsquigarrow$

$\rightsquigarrow$ $\rightsquigarrow$

[15] Hughes 2005

How to represent cut[15]

Combinatorial proofs allows to represent cut-free proofs

Fact

Proof of Γ *with a cut on a formula* $A \iff$ *Proof of* $\Gamma, A \wedge \bar{A}$

$\rightsquigarrow$ $\rightsquigarrow$

$\rightsquigarrow$ $\rightsquigarrow$

[15] Hughes 2005

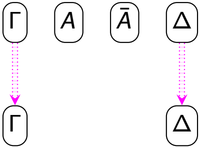
How to represent cut[15]

Combinatorial proofs allows to represent cut-free proofs

Fact

Proof of Γ *with a cut on a formula* $A \iff$ *Proof of* $\Gamma, A \wedge \bar{A}$

$$\text{cut}\,\frac{\Gamma, A \quad \bar{A}, \Delta}{\Gamma, \Delta} \quad \rightsquigarrow \qquad\qquad \rightsquigarrow$$



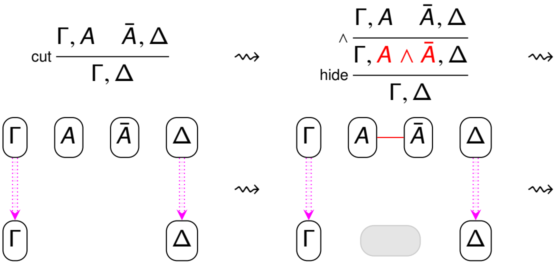
$\rightsquigarrow \qquad\qquad \rightsquigarrow$

[15] Hughes 2005

Combinatorial proofs allows to represent cut-free proofs

Fact

Proof of Γ with a cut on a formula $A \iff$ Proof of $\Gamma, A \wedge \bar{A}$

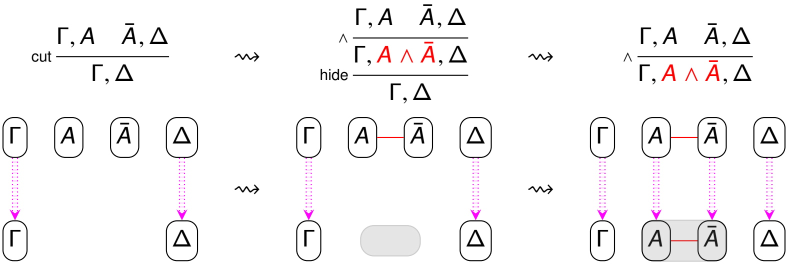


[15] Hughes 2005

Combinatorial proofs allows to represent cut-free proofs

Fact

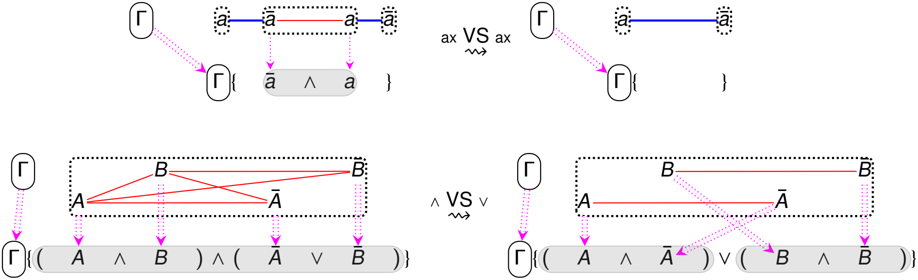
Proof of Γ with a cut on a formula A $\iff$ *Proof of $\Gamma, A \wedge \bar{A}$*



[15] Hughes 2005

Cut-elimination[16]

Cut-elimination = elimination of contradictions



[16] Acclavio TLLA2020

Cut-elimination[17]

A different approach:

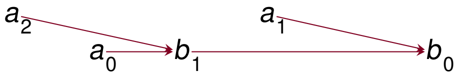
[17] Straßburger FSCD2017

Related and Future works

Proof certificates desiderata

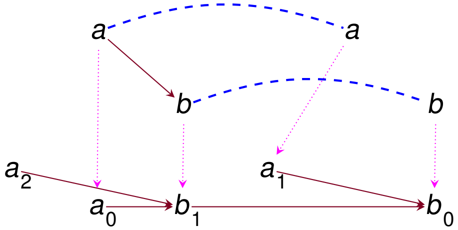
- A certificate contains all the information in a proof
- A certificate contains only the information in a proof
- A certificate can be checked in polynomial time if it is correct
- Certificates can be composed

There is a relation between ICPs and winning innocent strategies:



$$\mathsf{S} = \left\{ \begin{array}{c} b_0^\circ b_1^\bullet a_0^\circ a_1^\bullet \\ b_0^\circ b_1^\bullet a_2^\circ a_1^\bullet \end{array} \right\}$$

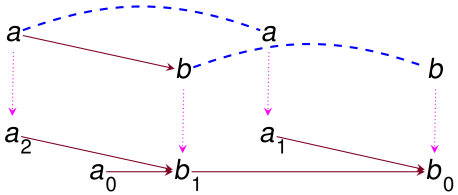
There is a relation between ICPs and winning innocent strategies:



$$\mathsf{S} = \left\{ \begin{array}{c} b_0^\circ b_1^\bullet a_0^\circ a_1^\bullet \\ b_0^\circ b_1^\bullet a_2^\circ a_1^\bullet \end{array} \right\} \quad \leftarrow$$

Game Semantics

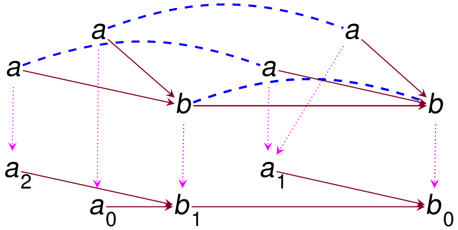
There is a relation between ICPs and winning innocent strategies:



$$S = \left\{ \begin{array}{c} b_0^\circ b_1^\bullet a_0^\circ a_1^\bullet \\ b_0^\circ b_1^\bullet a_2^\circ a_1^\bullet \end{array} \right\} \quad \leftarrow$$

Game Semantics

There is a relation between ICPs and winning innocent strategies:

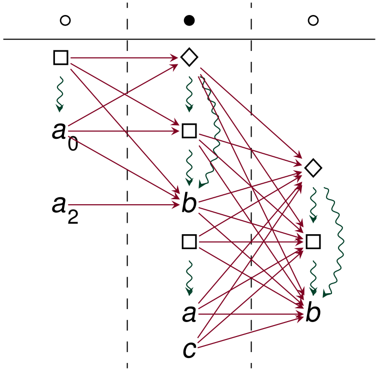


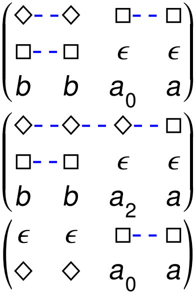
$$S = \left\{ \begin{array}{c} b_0^\circ b_1^\bullet a_0^\circ a_1^\bullet \\ b_0^\circ b_1^\bullet a_2^\circ a_1^\bullet \end{array} \right\}$$

This is an intuitionistic combinatorial proof!

New Game Semantics[18]

You can use combinatorial proofs to design game semantics





[18] Acclavio, Catta & Straßburger 2021

- Combinatorial proofs are a proof system
- Combinatorial proofs capture proof equivalence
- We have combinatorial proofs for different logics

What next?

- More combinatorial proofs !
- Combinatorial proofs compositionality
- Implement proof certificates

Thank you

Questions?