The proof theory of pomsets

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   - Coherent Spaces
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From Linear Logic to Pomset Logic

Linear Logic[1]

[1] Girard 1989

Pomset logic

Pomset formulas

$$A, B ::= a \mid a^{\perp} \mid \underbrace{A \parr B \mid A \oplus B}_{\text{disjunction}} \mid \underbrace{A \otimes B \mid A \& B}_{\text{conjunction}} \mid ?A \mid !A$$

Pomset logic

Pomset formulas

$$A, B ::= a \mid a^\perp \mid \underbrace{A \parr B \mid A \oplus B}_{\text{disjunction}} \mid \underbrace{A \otimes B \mid A \& B}_{\text{conjunction}} \mid ?A \mid !A$$

Negation $(\cdot)^\perp$ such that:

$$A^{\perp\perp} = A \qquad (A \parr B)^\perp = A^\perp \otimes B^\perp \qquad (A \oplus B)^\perp = A^\perp \& B^\perp$$

(Linear) **implication** defined "classically" (via multiplicative):

$$A \multimap B := A^\perp \parr B$$

Pomset logic

Pomset formulas

$$A, B ::= a \mid a^{\perp} \mid \underbrace{A \parr B \mid A \oplus B}_{\text{disjunction}} \mid \underbrace{A \otimes B \mid A \& B}_{\text{conjunction}} \mid ?A \mid !A$$

Negation $(\cdot)^{\perp}$ such that:

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(Linear) **implication** defined "classically" (via multiplicative):

$$A \multimap B := A^{\perp} \parr B$$

Sequent calculus rules

$$\text{ax}\ \frac{}{\vdash a, a^{\perp}} \qquad \parr\ \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \otimes\ \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \quad \text{multiplicative}$$

$$\text{ax}\ \frac{\vdash \Gamma, A_i}{\vdash \Gamma, A_1 \parr A_2} \qquad \&\ \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \quad \text{multiplicative}$$

Coherent Spaces[2]

[2] Kind of Girard 1989

Pomset logic

Coherent space = countable set + binary reflexive symmetric relation

$$\mathfrak{C} = \langle |\mathfrak{C}|, \coh \rangle$$

Additional relations

$$\begin{array}{lcl} a \frown b := a \coh b & \text{and} & a \neq b \\ a \asymp b := a \not\coh b & \text{or} & a = b \\ a \smile b := a \asymp b & \text{and} & a \neq b \end{array}$$

Theorem

Coherent spaces are a denotational semantics for linear logic.

Coherent space = countable set + binary reflexive symmetric relation

$G = \langle V_G, \frown\!\!\!\!\smile \rangle$ (i.e., a graph with countable vertices)

Additional relations

$$
\begin{array}{lcl}
a \frown b := a \,\frown\!\!\!\!\smile\, b & \text{and} & a \neq b \\
a \asymp b := a \,\not\frown\!\!\!\!\smile\, b & \text{or} & a = b \\
a \smile b := a \asymp b & \text{and} & a \neq b
\end{array}
$$

Theorem

Coherent spaces are a denotational semantics for linear logic.

The non-commutative connective $\lessdot$[3]

[3] Retoré 1993

Pomset logic

Let G and H be coherent spaces:

	$\{\!\{A \parr B\}\!\}$		$\{\!\{A \otimes B\}\!\}$
vertices	$V_G \times V_H$		
edges	(see table below)		(see table below)

Edges of $\{\!\{A \parr B\}\!\}$:

$A \backslash B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

Edges of $\{\!\{A \otimes B\}\!\}$:

$A \backslash B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

	$\{\!\{G \oplus H\}\!\}$	$\{\!\{G \,\&\, H\}\!\}$
vertices	$V_G \uplus V_H$	
edges	$\supset_G \cup \supset_H$	$\supset_G \cup \supset_H \cup \{(x, y) \mid x \in V_G, y \in V_H\}$

save for later

Pomset logic

Let G and H be coherent spaces:

	$\{\!\{A \parr B\}\!\}$	$\{\!\{A \lhd B\}\!\}$	$\{\!\{A \otimes B\}\!\}$
vertices	$V_G \times V_H$		
edges	(see below)	(see below)	(see below)

Edges of $\{\!\{A \parr B\}\!\}$:

$A \backslash B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

Edges of $\{\!\{A \lhd B\}\!\}$:

$A \backslash B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

Edges of $\{\!\{A \otimes B\}\!\}$:

$A \backslash B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

Pomsets and Pomset logic

Pomset formulas and SP-orders[4]

[4] Retoré 1993, Guglielmi 2007

Pomset formulas:

$$A, B ::= \varnothing \mid a \mid a^{\perp} \mid A \parr B \mid A \lhd B \mid A \otimes B$$

Dicographs[5] or Relation webs[6] = graphs encoding Pomset formulas

[Generalization of results on cographs from ’60s]

[5] Retoré ’93

[6] Guglielmi ’07

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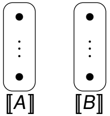
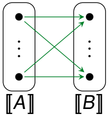
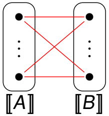
[Generalization of results on cographs from ’60s]

$[\![\varnothing]\!]$	$[\![a]\!]$	$[\![a^{\perp}]\!]$	$[\![A \parr B]\!] = [\![A]\!] \parr [\![B]\!]$	$[\![A \lhd B]\!] = [\![A]\!] \lhd [\![B]\!]$	$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$
	a	$a^{\perp}$			

[5] Retoré ’93
[6] Guglielmi ’07

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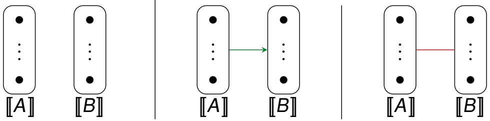
$$A, B ::= \varnothing \mid a \mid a^{\perp} \mid A \parr B \mid A \lhd B \mid A \otimes B$$

Dicographs[5] or Relation webs[6] = graphs encoding Pomset formulas

[Generalization of results on cographs from '60s]

$[\![\varnothing]\!]$	$[\![a]\!]$	$[\![a^{\perp}]\!]$	$[\![A \parr B]\!] = [\![A]\!] \parr [\![B]\!]$	$[\![A \lhd B]\!] = [\![A]\!] \lhd [\![B]\!]$	$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$
	a	$a^{\perp}$			



[5] Retoré '93

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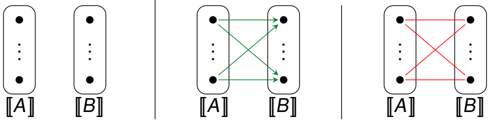
Pomset formulas:

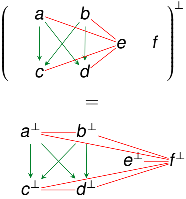
$$A, B ::= \varnothing \mid a \mid a^\perp \mid A \parr B \mid A \triangleleft B \mid A \otimes B$$

Dicographs[5] or Relation webs[6] = graphs encoding Pomset formulas

[Generalization of results on cographs from '60s]

$[\![\varnothing]\!]$	$[\![a]\!]$	$[\![a^\perp]\!]$	$[\![A \parr B]\!] = [\![A]\!] \parr [\![B]\!]$	$[\![A \triangleleft B]\!] = [\![A]\!] \triangleleft [\![B]\!]$	$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$
	a	$a^\perp$			





[5] Retoré '93

[6] Guglielmi '07

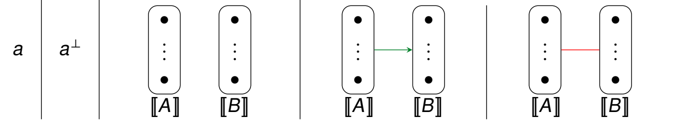
Pomset formulas:

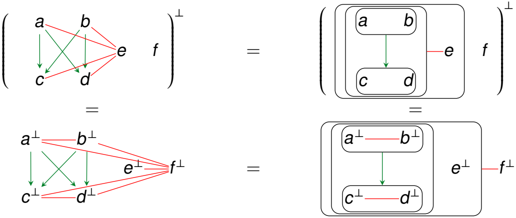
$$A, B ::= \varnothing \mid a \mid a^{\perp} \mid A \parr B \mid A \lhd B \mid A \otimes B$$

Dicographs[5] or Relation webs[6] = graphs encoding Pomset formulas

[Generalization of results on cographs from '60s]

$$[\![\varnothing]\!] \quad [\![a]\!] \quad [\![a^{\perp}]\!] \quad [\![A \parr B]\!] = [\![A]\!] \parr [\![B]\!] \quad [\![A \lhd B]\!] = [\![A]\!] \lhd [\![B]\!] \quad [\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$$

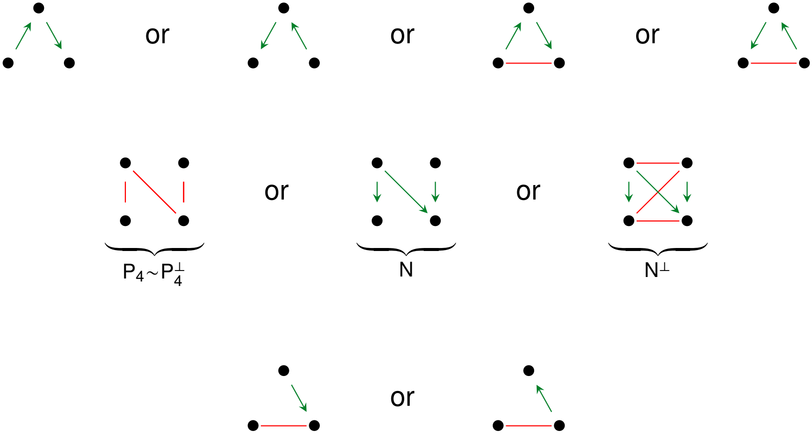




[5] Retoré '93

[6] Guglielmi '07

A graph containing one of the following induced subgraph cannot be a $[\![F]\!]$



Soundness of implication in Pomset[7]

[7] Retoré 1993, Bechet & de Groote & Retoré 1997

Theorem (Transitivity of $\multimap$)

If $\vdash_{\text{Pomset}} A \multimap B$ and $\vdash_{\text{Pomset}} B \multimap C$, then $\vdash_{\text{Pomset}} A \multimap C$.

Theorem

*An order O is **series-parallel** iff there is a Pomset-formula A (containing no $\otimes$) such that $[\![A]\!] = O$.*

Theorem

*Let $[\![A]\!]$ and $[\![B]\!]$ be two **series-parallel** orders.*
If $[\![A]\!] \supseteq [\![B]\!]$ then $\vdash_{\text{Pomset}} A \multimap B$.

Sketch of proof.

In Bechet & de Groote & Retoré 1997 is provided a complete system of rewriting rules to characterize SP-orders inclusion.
Each rewriting rule $\{\!\{A\}\!\} \rightarrow \{\!\{B\}\!\}$ corresponds to an implication $A \multimap B$, which is valid in Pomset. Thus we conclude by transitivity of $\multimap$. □

Pomset Logic offspring

Logic	Formalism for proofs	Complexity (provability)
Pomset	Proof nets[8]	unknown until 2020

[8] Retoré 1993 and 1996
[9] Guglielmi 2007
[10] Kahramanoğullari 2008

Logic	Formalism for proofs	Complexity (provability)
Pomset	Proof nets[8]	unknown until 2020

BV-formulas = Pomset-formulas

Logic	Formalism for proofs	Complexity (provability)
BV	Deep Inference[9]	NP[10]

[8] Retoré 1993 and 1996

[9] Guglielmi 2007

[10] Kahramanoğullari 2008

Deep inference (and why it is needed)[11]

[11] Guglielmi 2007, Tiu 2006

$$6 \times (3 + 4)$$

$6 \times (3 + 4)$

6×7

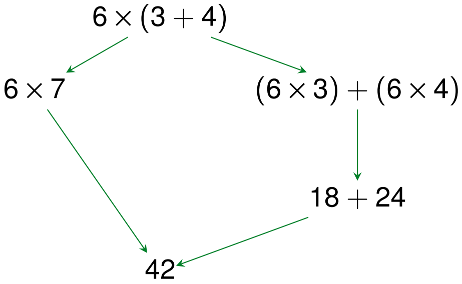
42

$$6 \times (3 + 4)$$

$$(6 \times 3) + (6 \times 4)$$

$$18 + 24$$

$$42$$


$6 \times (3 + 4)$
6×7
$(6 \times 3) + (6 \times 4)$
$18 + 24$
42

$$\textit{mult}\;\dfrac{\dfrac{\dfrac{6\times(3+4)}{6\times\left(\textit{sum}\;\dfrac{3+4}{7}\right)}}{6\times 7}}{42}$$

$$\textit{dist}\;\dfrac{6\times(3+4)}{\dfrac{\left(\textit{mult}\;\dfrac{\dfrac{6\times 3}{3\times 6}}{18}\right)+\left(\textit{mult}\;\dfrac{6\times 4}{24}\right)}{\textit{sum}\;\dfrac{18+24}{42}}}$$

The same in deep inference:

- $G \implies \begin{array}{c} G \\ \mathsf{Id}_G \| \\ G \end{array} \quad \left(\text{or } \begin{array}{c} G' \\ \cdots\cdots \\ G \end{array} \text{ if } G \equiv G' \right)$

The same in deep inference:

- $G \quad \Longrightarrow \quad \begin{array}{c} G \\ \mathrm{Id}_G \, \| \\ G \end{array} \qquad \left(\text{or } \begin{array}{c} G' \\ \cdots\cdots \\ G \end{array} \text{ if } G \equiv G' \right)$

- $\begin{array}{c} H_1 \\ \mathcal{D}_1 \, \| \\ G_1 \end{array}$ and $\begin{array}{c} H_2 \\ \mathcal{D}_2 \, \| \\ G_2 \end{array}$ and $\mathit{rule} \, \dfrac{G_1}{H_2} \quad \Longrightarrow \quad \mathit{rule} \, \dfrac{\begin{array}{c} H_1 \\ \mathcal{D}_1 \, \| \\ G_1 \end{array}}{\begin{array}{c} H_2 \\ \mathcal{D}_2 \, \| \\ G_2 \end{array}}$

The same in deep inference:

- $G \Longrightarrow \begin{array}{c} G \\ \mathrm{Id}_G \, \| \\ G \end{array} \quad \left(\text{or } \begin{array}{c} G' \\ \vdots \\ G \end{array} \text{ if } G \equiv G' \right)$

- $\begin{array}{c} H_1 \\ \mathcal{D}_1 \, \| \\ G_1 \end{array}$ and $\begin{array}{c} H_2 \\ \mathcal{D}_2 \, \| \\ G_2 \end{array}$ and $\mathit{rule}\,\dfrac{G_1}{H_2} \Longrightarrow \mathit{rule}\,\dfrac{\begin{array}{c} H_1 \\ \mathcal{D}_1 \, \| \\ G_1 \end{array}}{\begin{array}{c} H_2 \\ \mathcal{D}_2 \, \| \\ G_2 \end{array}}$

- $\begin{array}{c} H_i \\ \mathcal{D}_i \, \| \\ G_i \end{array}$ and P an n-ary term-constructor $\Longrightarrow P \left(\begin{array}{c} H_1 \\ \mathcal{D}_1 \, \| \\ G_1 \end{array}, \ldots, \begin{array}{c} H_n \\ \mathcal{D}_n \, \| \\ G_n \end{array} \right)$

A deep inference system for BV

$$\mathsf{ai}\!\downarrow \frac{\varnothing}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{q}\!\downarrow \frac{(A \parr C) \lhd (B \parr D)}{(A \lhd B) \parr (C \lhd D)}$$

F is provable (denoted $\vdash_{\mathsf{BV}} F$) if there is a $\begin{array}{c}\varnothing\\ \mathcal{D}\,\Vert\,\mathsf{BV}\\ F\end{array}$

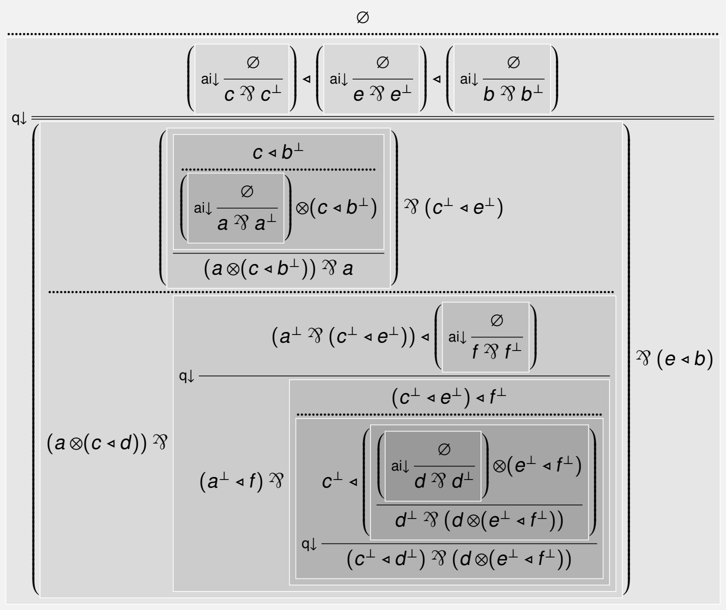
Formulas are considered modulo the following equivalences:

Associativity	Unit	Commutativity ($\parr$ and $\otimes$)
$A \parr (B \parr C) \equiv (A \parr B) \parr C$	$A \parr \varnothing \equiv A$	$A \parr B \equiv B \parr A$
$A \lhd (B \lhd C) \equiv (A \lhd B) \lhd C$	$A \lhd \varnothing \equiv A \equiv \varnothing \lhd A$	
$A \otimes (B \otimes C) \equiv (A \otimes B) \otimes C$	$A \otimes \varnothing \equiv A$	$A \otimes B \equiv B \otimes A$

Example: a proof of $(a^{\perp} \triangleleft (b^{\perp} \parr c^{\perp})) \parr ((a \parr b) \triangleleft c)$

$$
\dfrac{\emptyset}{\text{q}\!\downarrow\dfrac{\left(\text{q}\!\downarrow\dfrac{\left(\text{ai}\!\downarrow\dfrac{\emptyset}{a \parr a^{\perp}}\right) \triangleleft \left(\text{ai}\!\downarrow\dfrac{\emptyset}{b \parr b^{\perp}}\right)}{(a^{\perp} \triangleleft b^{\perp}) \parr (a \triangleleft b)}\right) \triangleleft \left(\text{ai}\!\downarrow\dfrac{\emptyset}{c \parr c^{\perp}}\right)}{\dfrac{(a^{\perp} \triangleleft b^{\perp}) \triangleleft c^{\perp}}{\left(a^{\perp} \triangleleft \left(\text{q}\!\downarrow\dfrac{\left(\dfrac{b^{\perp}}{b^{\perp} \parr \emptyset}\right) \triangleleft \left(\dfrac{c^{\perp}}{\emptyset \parr c^{\perp}}\right)}{\left(\dfrac{b^{\perp} \triangleleft \emptyset}{b^{\perp}}\right) \parr \left(\dfrac{c^{\perp} \triangleleft \emptyset}{c^{\perp}}\right)}\right)\right)} \parr \left(\left(\text{q}\!\downarrow\dfrac{\left(\dfrac{a}{a \parr \emptyset}\right) \triangleleft \left(\dfrac{b}{\emptyset \parr b}\right)}{\left(\dfrac{a \triangleleft \emptyset}{a}\right) \parr \left(\dfrac{b \triangleleft \emptyset}{b}\right)}\right) \triangleleft c\right)}}
$$

Deep inference is needed!



Pomset $\neq$ BV[12]

[12] Nguyên & Strassburger 2022

Logic	Formalism for proofs	Complexity (provability)
Pomset	Proof nets[13]	Σ_2^P [14]
BV	Deep Inference[15]	NP[16]

The following formula is provable in Pomset but not in BV

$$((a \triangleleft b) \otimes (c \triangleleft d)) \parr ((e \triangleleft f) \otimes (g \triangleleft h)) \parr (a^\perp \triangleleft h^\perp) \parr (e^\perp \triangleleft b^\perp) \parr (c^\perp \triangleleft f^\perp) \parr (g^\perp \triangleleft d^\perp)$$

Theorem (Transitivity of $\multimap$)

If $\vdash_{\mathrm{BV}} A \multimap B$ and $\vdash_{\mathrm{BV}} B \multimap C$, then $\vdash_{\mathrm{BV}} A \multimap C$.

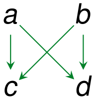
Theorem

Let $[\![A]\!]$ and $[\![B]\!]$ be two ***series-parallel*** *orders.*
If $[\![A]\!] \supseteq [\![B]\!]$ then $\vdash_{\mathrm{BV}} A \multimap B$.

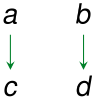
[13] Retoré 1993 and 1996
[14] Nguyên 2020
[15] Guglielmi 2007, Tiu2006
[16] Kahramanoğullari 2008

The graphical logic GV[17]

[17] A. & Horne & Mauw & Straßburger 2022

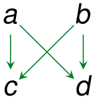


$\multimap$

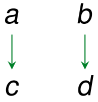


Provable in BV

$$(a \parr b) \lhd (c \parr d) \multimap (a \lhd c) \parr (b \lhd d)$$



$\multimap$



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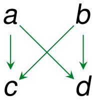
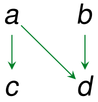
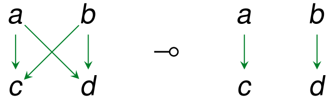


$\multimap$

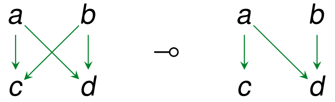


$(a \parr b) \lhd (c \parr d) \multimap$ NOT A FORMULA



$$(a \parr b) \lhd (c \parr d) \multimap (a \lhd c) \parr (b \lhd d)$$

Provable in BV
(series-parallel orders)



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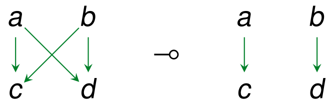
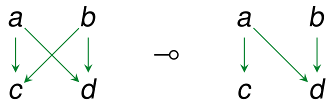


$$(a \parr b) \lhd (c \parr d) \multimap (a \lhd c) \parr (b \lhd d)$$

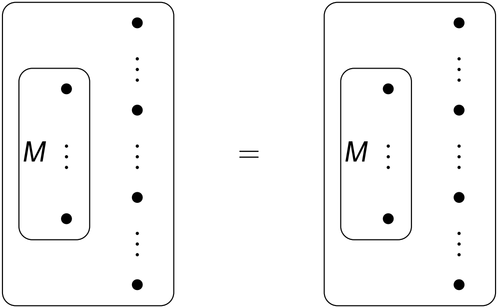
Provable in BV
(series-parallel orders)



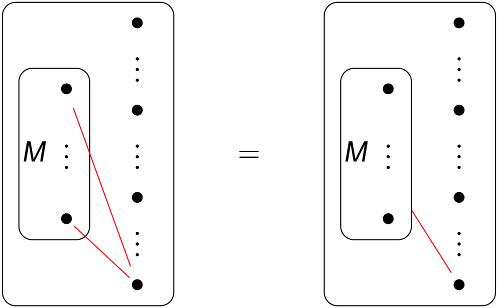
$(a \parr b) \lhd (c \parr d) \multimap$ NOT A FORMULA

???

A ***module*** of a graph $G = H[M]$ is a set of vertices M s.t.



A ***module*** of a graph $G = H[M]$ is a set of vertices M s.t.



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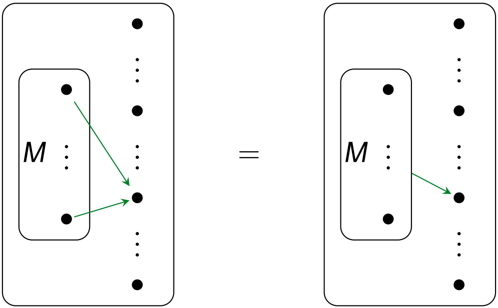

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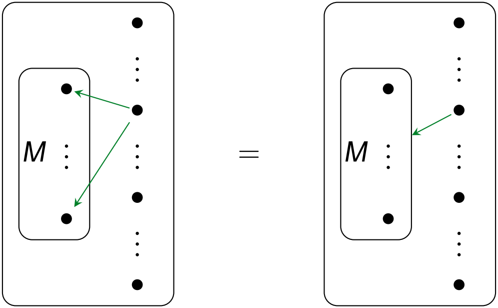

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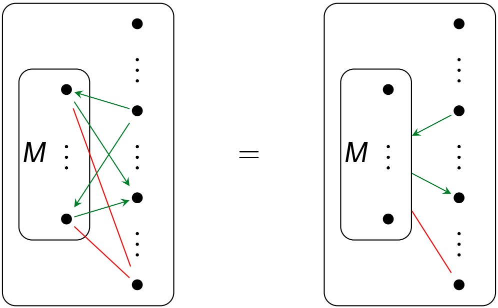

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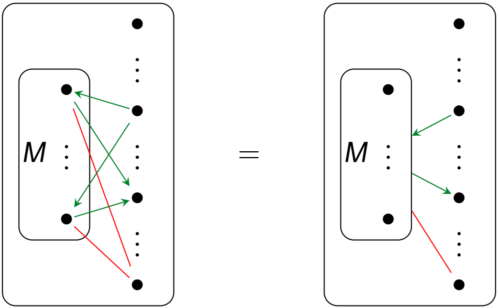
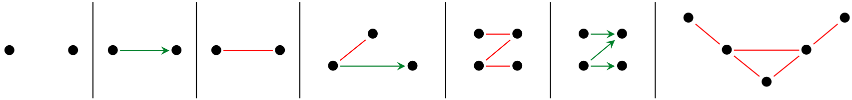


A graph G is ***prime*** if it has modules V_G, $\varnothing$ and $\{x\}$ for all $x \in V_G$.



If G has n vertices and $H_1, \ldots, H_n$ graphs,
then we use G as a **logic connective** and we write $G\llparenthesis H_1, \ldots, H_n \rrparenthesis$

$\parr$: 	$\lhd$: 	$\otimes$: 
$\parr\llparenthesis G, H \rrparenthesis = G \parr H$	$\lhd\llparenthesis G, H \rrparenthesis = G \lhd H$	$\otimes\llparenthesis G, H \rrparenthesis = G \otimes H$

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$\parr$: • •	$\triangleleft$: • → •	$\otimes$: • — •
$\parr(\!|G, H|\!) = G \parr H$	$\triangleleft(\!|G, H|\!) = G \triangleleft H$	$\otimes(\!|G, H|\!) = G \otimes H$

Lemma (Modular decomposition of graphs (Gallai '75))

If $G \neq \emptyset$ is a graph, then we have exactly one of the following cases:

(i) *G is a singleton graph*

(ii) *$G = P(\!|A_1, \ldots, A_n|\!)$ for a prime graph P*

GS

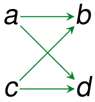
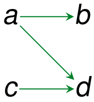
$$\mathsf{ai}\!\downarrow \frac{\emptyset}{a^\perp \parr a} \qquad \mathsf{p}\!\downarrow \frac{(\mathbf{M_1} \parr N_1) \otimes \cdots \otimes (\mathbf{M_n} \parr N_n)}{R^\perp (\!|\mathbf{M_1}, \ldots, \mathbf{M_n}|\!) \parr R(\!|N_1, \ldots, N_n|\!)}$$

$$\mathsf{s}_\parr \frac{P(\!|M_1, \ldots, M_{i-1}, \mathbf{M_i} \parr N, M_{i+1}, \ldots M_n|\!)}{\mathbf{M_i} \parr P(\!|M_1, \ldots, M_{i-1}, N, M_{i+1}, \ \ldots, M_n|\!)} \qquad \mathsf{s}_\otimes \frac{\mathbf{M_i} \otimes P(\!|M_1, \ldots, M_{i-1}, N, M_{i+1}, \ldots, M_n|\!)}{P(\!|M_1, \ldots, M_{i-1}, \mathbf{M_i} \otimes N, M_{i+1}, \ldots, M_n|\!)}$$

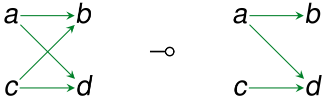
$$\mathsf{q}\!\downarrow \frac{Q^\perp(\!|\mathbf{M_1} \parr N_1, \ldots, \mathbf{M_n} \parr N_n|\!)}{Q^\perp(\!|\mathbf{M_1}, \ldots, \mathbf{M_n}|\!) \parr Q(\!|N_1, \ldots, N_n|\!)} \qquad \mathsf{qm} \frac{Q(\!|\mathbf{M_1} \parr N_1, \ldots, \mathbf{M_n} \parr N_n|\!)}{Q(\!|\mathbf{M_1}, \ldots, \mathbf{M_n}|\!) \parr Q(\!|N_1, \ldots, N_n|\!)}$$

$$\mathsf{sl} \frac{Q(\!|M_1, \ldots, M_k, \emptyset, \ldots, \emptyset|\!) \triangleleft Q(\!|\emptyset, \ldots, \emptyset, M_{k+1}, \ldots, M_n|\!)}{Q(\!|M_1, \ldots, M_n|\!)}$$

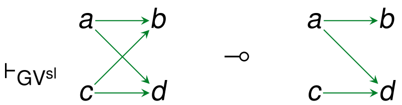
$$\mathsf{GV} = \mathsf{GS} \cup \{\mathsf{q}\!\downarrow, \mathsf{qm}\} \qquad \mathsf{GV}^{\mathsf{sl}} = \mathsf{GV} \cup \{\mathsf{sl}\}$$

$\vdash_{\mathsf{GV}^{\mathsf{sl}}}$  $\multimap$ 

$\vdash_{GV^{sl}}$



Idea: the rule sl allows linearization (see Lamport logical clocks)



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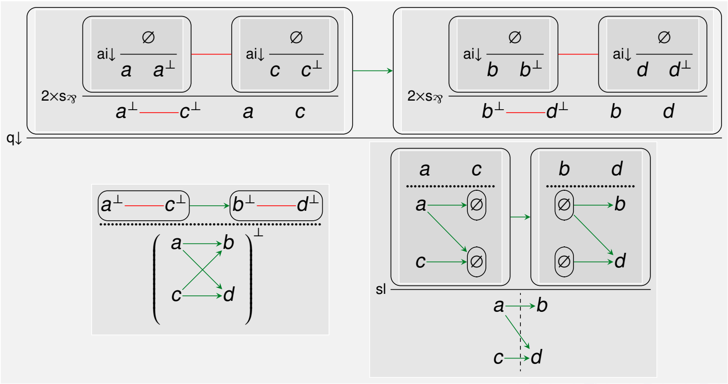

Theorem

GS, GV *and* $\mathsf{GV^{sl}}$ *are proof systems [in the sense of Cook-Reckhow]*

Theorem

Proofs in $\mathsf{GV^{sl}}$ *have polynomial size. Therefore provability is in* NP.

Theorem (Transitivity of $\multimap$)

Let $\mathsf{GX} \in \{\mathsf{GS}, \mathsf{GV}, \mathsf{GV^{sl}}\}$. *If* $\vdash_{\mathsf{GX}} A \multimap B$ *and* $\vdash_{\mathsf{GX}} B \multimap C$, *then* $\vdash_{\mathsf{GX}} A \multimap C$.

Theorem

- GS *is a conservative extension of* MLL°
- GV *and* $\mathsf{GV^{sl}}$ *are conservative extensions of both* GS *and* BV

Other extensions and applications

- Retoré 1999: Petri net execution as cut-elimination in Pomset [18]
- Bruscoli 2002: BV and (a minimal) CCS:
- Horne 2015: extending the work of Bruscoli with additive connectives (model choice/conflict relation)
- Brunet & Pym 2019 (unconscious child): “pomset with boxes”[19]
- Horne & Tiu 2019: nominal quantifiers (model fresh names)
- Guglielmi/Reddy/Retore (alphabetic order): modality for the $\triangleleft$ fixpoint ($\daleth A \equiv A \triangleleft \daleth A$)

[18] Does the proofs in Brunet & Pous & Struth 2017 use the same technique/encoding?
[19] My conjecture is that it's just BV + a modal operator

Process calculi and BV[20]

20

A minimal CCS

Processes: : $p, q := \mathrm{NIL} \mid \alpha.p \mid \bar{\alpha}.p \mid p \parallel q \mid$ with $\alpha \in \mathcal{A}$

Communication rule: $$\frac{}{a.p \parallel \bar{a}.q \ \longrightarrow_a \ p \parallel q}$$

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Communication rule: $$\frac{}{a.p \parallel \bar{a}.q \longrightarrow_a p \parallel q}$$

Translation:

$$\{\!\{\mathrm{NIL}\}\!\} = \circ \quad \{\!\{\alpha.p\}\!\} = a_\alpha \triangleleft \{\!\{p\}\!\} \quad \{\!\{\bar{\alpha}.p\}\!\} = a_\alpha^\perp \triangleleft \{\!\{p\}\!\} \quad \{\!\{p \parallel q\}\!\} = \{\!\{p\}\!\} \parr \{\!\{q\}\!\}$$

Theorem (Bruscoli 2002)

There is a terminating execution of p $\iff \vdash_{\mathsf{BV}^L} \{\!\{p\}\!\}$

The trace $a_{\alpha_1}, \ldots, a_{\alpha_n}$ *is valid for p* $\iff \vdash_{\mathsf{BV}^L} (a_{\alpha_1} \triangleleft \cdots \triangleleft a_{\alpha_n}) \multimap \{\!\{p\}\!\}$

Extensions handling:

- choice operator (Horne 2015)
- nominal quantifiers (Horne& Tiu)
- recursion using the modality ʔ ?

Future works:

(-) Categorical and Algebraic Semantics

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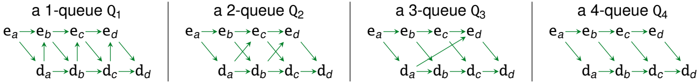

$$\vdash_{BV} Q_1 \multimap Q_2 \quad \vdash_{GV^{sl}} Q_2 \multimap Q_3 \quad \vdash_{???} Q_3 \multimap Q_4$$

- (-) Proof system operating on graph decomposition via interfaces (e.g., ipomset Fahrenberga & alt.) instead of modular decomposition?

Thank you

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Questions?
Comments?
Feedbacks?