Game Semantics for Constructive Modal Logic

Matteo Acclavio

Università Roma Tre

(Joint works with Davide Catta and Lutz Strassburger)

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1 Motivations

2 Game Semantics for Intuitionistic Logic
- Intuitionistic Logic
- Arenas (the game boards)
- How to play: A crush curse in Game Semantics
- From Strategies to Combinatorial Proofs

3 Game Semantics for Constructive Modal Logic
- What is Constructive Modal Logic?
- From Combinatorial Proofs to Strategies

What is a proof?

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- A strategy to win an argument on the truthful of a statement

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What is a proof?

- A strategy to win an argument on the truthful of a statement
- A sequence of instruction to reduce a statement to axioms
- A collection of the interactions between the components of a statement

$$[\![-]\!]: \quad \{\text{ Proofs }\} \quad \to \quad \{\text{ Denotations }\}$$

+

Compositionality

+

$$\mathfrak{D} \rightsquigarrow^*_{\text{cut}} \mathfrak{D}' \quad \Rightarrow \quad [\![\mathfrak{D}]\!] = [\![\mathfrak{D}']\!]$$

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Denotational semantics for constructive modal logic(s)
[Bellin, De Paiva and Ritter, 2001]

$$\frac{\left\{\lambda\text{-terms}\right\}}{\beta\text{-reduction}}$$

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We want more! Let's do a game semantics!

Propositional Intuitionistic Logic Formulas

$$A, B ::= \mathsf{1} \mid a \mid A \supset B \mid A \wedge B$$

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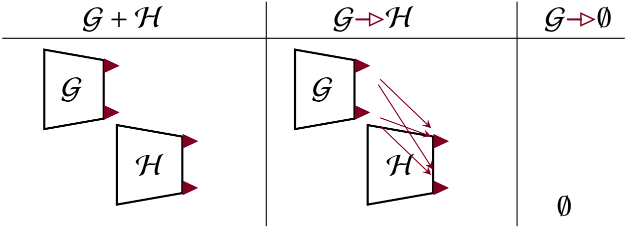
$$A, B ::= \mathsf{1} \mid a \mid A \supset B \mid A \wedge B$$

. . . enough for λ-calculus with pairs

Arenas:

$$[\![a]\!] = a \qquad [\![1]\!] = \emptyset \qquad [\![A \wedge B]\!] = [\![A]\!] + [\![B]\!] \qquad [\![A \supset B]\!] = [\![A]\!] \mathrel{-\!\!\triangleright} [\![B]\!]$$



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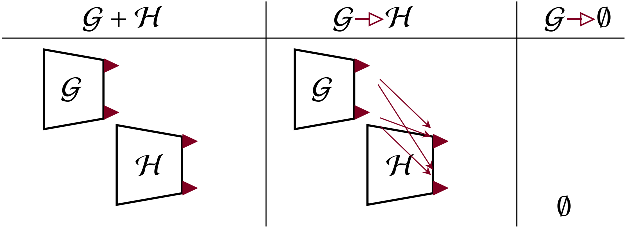


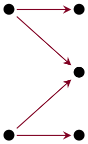
Examples:

$$[\![((b_1 \supset b_0) \supset a_1) \supset (a_2 \wedge a_0)]\!] =$$

$$[\![((a \wedge a) \supset b) \supset (a \supset b)]\!] =$$

Theorem

A direct acyclic graph $\mathcal{G}$ is an arena iff it contains no induced



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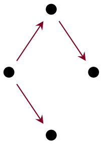

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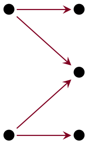

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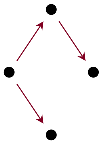

Theorem

$[\![A \wedge (B \wedge C)]\!] = [\![(A \wedge B) \wedge C]\!]$ *and* $[\![A \multimap (B \multimap C)]\!] = [\![(A \wedge B) \multimap C]\!]$

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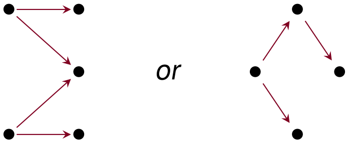

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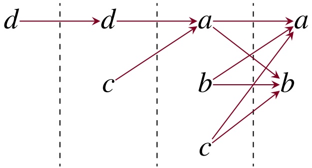
Lemma

Arenas are stratified.

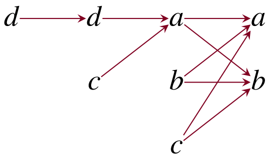


How to play:

A crush curse in Game Semantics

How to play:

- Two-players game (∘ and •)
- ∘ starts on a root
- each non initial move is *justified* ($\rightarrow$) by <u>one</u> previous move
- each •-move must “reply” to the previous ∘-move (same label)
- ∘-moves are justified by <u>the</u> previous •-move (∘ is *shortsighted*)
- a player wins when the other is out of moves



How to play:

- Two-players game ($\circ$ and $\bullet$)
- $\circ$ starts on a root
- each non initial move is *justified* ($\rightarrow$) by one previous move
- each $\bullet$-move must "reply" to the previous $\circ$-move (same label)
- $\circ$-moves are justified by the previous $\bullet$-move ($\circ$ is *shortsighted*)
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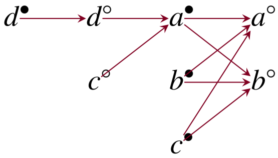


“A strategy to win an argument on the truthful of a statement”

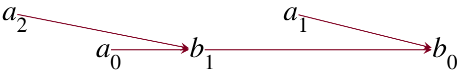
- Play: sequence of moves

"A strategy to win an argument on the truthful of a statement"

- Play: sequence of moves
- WIS (for •): predecessor-closed set of plays
    - taking into account every possible ∘-move
    - maximal plays always end with •-moves

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

It is ∘'s turn



$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ \\ \\ \\ \end{array} \right\}$$

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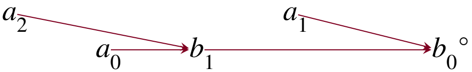
It is $\bullet$'s turn



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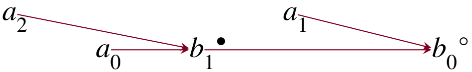

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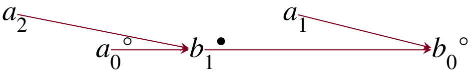

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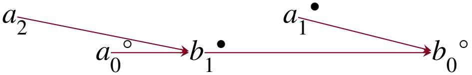
It is ∘'s turn

PLAYER • WINS!



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It is $\circ$'s turn

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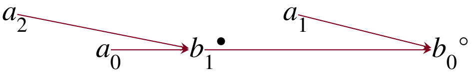

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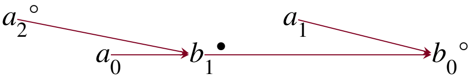

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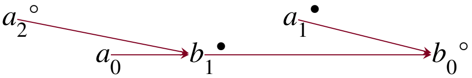

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$$[\![-]\!] : \quad \{ \text{Proofs in LI} \} \quad \rightarrow \quad \{ \text{Winning innocent strategies} \}$$

$$\mathfrak{D}_F \quad \rightarrow \quad \text{winning strategy on } [\![F]\!]$$

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$$\frac{}{a \vdash a}\,\mathsf{AX} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\supset^{\mathsf{R}} \quad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\supset^{\mathsf{L}} \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\wedge^{\mathsf{R}} \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\wedge^{\mathsf{L}}$$

$$\frac{}{\vdash 1}\,1^R \quad \frac{\Gamma \vdash A}{\Gamma, 1 \vdash A}\,1^L \quad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathsf{C} \quad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathsf{W}$$

From a LI-proof of F to a WIS on $[\![F]\!]$:

$$\left[\!\!\left[\frac{}{a^\bullet \vdash a^\circ}\ \mathsf{AX}\right]\!\!\right] = \{\epsilon, a^\circ, a^\circ a^\bullet\} \quad \left[\!\!\left[\frac{}{\vdash 1^\circ}\ 1^R\right]\!\!\right] = \{\epsilon\}$$

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From a LI-proof of F to a WIS on $[\![F]\!]$:

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From a LI-proof of F to a WIS on $[\![F]\!]$:

$$\left[\!\!\left[\frac{}{a^\bullet \vdash a^\circ}\,\mathsf{AX}\right]\!\!\right] = \{\epsilon, a^\circ, a^\circ a^\bullet\} \qquad \left[\!\!\left[\frac{}{\vdash 1^\circ}\,1^R\right]\!\!\right] = \{\epsilon\}$$

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From a WIS **on** $[\![F]\!]$ **to a** LI**-proof of** F**:**

Sequent	Shape of $\mathcal{S}$	Shape of $\mathfrak{D}_{\mathcal{S}}$
$\vdash 1$	$\mathcal{S} = \{\epsilon\}$	$\dfrac{}{\vdash 1}\,1$
$a \vdash a$	$\mathcal{S} = \{\epsilon, a, aa\}$	$\dfrac{}{a \vdash a}\,\mathsf{AX}$
$\Gamma, B \wedge C \vdash A$	any	$\dfrac{\overset{\mathfrak{D}_{\mathcal{S}}\Vert}{\Gamma, B, C \vdash A}}{\Gamma, B \wedge C \vdash A}\,\wedge^{\mathsf{L}}$
$\Gamma \vdash B \supset A$	any	$\dfrac{\overset{\mathfrak{D}_{\mathcal{S}}\Vert}{\Gamma, B \vdash A}}{\Gamma \vdash B \supset A}\,\supset^{\mathsf{R}}$
$\Gamma \vdash A_1 \wedge A_2$ Γ contains no $\wedge$-formula	$\mathcal{S} = \mathcal{T} \cup \mathcal{R}$ $\mathcal{T} = \{\tau \in \mathcal{S} \mid \tau \text{ contains no moves in } A_2\}$ $\mathcal{R} = \{\rho \in \mathcal{S} \mid \rho \text{ contains no moves in } A_1\}$	$\dfrac{\dfrac{\overset{\mathfrak{D}_{\mathcal{T}}\Vert}{\Gamma \vdash A_1} \quad \overset{\mathfrak{D}_{\mathcal{R}}\Vert}{\Gamma \vdash A_2}}{\Gamma, \Gamma \vdash A_1 \wedge A_2}\,\wedge^{\mathsf{R}}}{\Gamma \vdash A_1 \wedge A_2}\,\mathsf{C}$
$\Gamma, A \supset B\{c^\bullet\} \vdash c^\circ$ c atomic and $A \supset B\{c^\bullet\} \neq c^\bullet$ $B\{c^\bullet\}$ contains the atom $c^\bullet$ Γ contains no $\wedge$-formulas	$c^\circ c^\bullet \in \mathcal{S}$ $\mathcal{T} = \{\tau \mid \text{there are } \sigma \text{ and } \tau' \text{ such that } \sigma\tau\tau' \in \mathsf{Split}^A_{\mathcal{S}}\}$ $\mathcal{R} = \{\rho \mid \text{there is no } \sigma \text{ such that } \rho\sigma \in \mathsf{Split}^A_{\mathcal{S}}\}$	$\dfrac{\dfrac{\overset{\mathfrak{D}_{\mathcal{T}}\Vert}{\Gamma \vdash A} \quad \overset{\mathfrak{D}_{\mathcal{R}}\Vert}{\Gamma, A \supset B\{c^\bullet\}, B\{c^\bullet\} \vdash c^\circ}}{\Gamma, \Gamma, A \supset B\{c^\bullet\}, A \supset B\{c^\bullet\} \vdash c^\circ}\,\supset^{\mathsf{L}}}{\Gamma, A \supset B\{c^\bullet\} \vdash c^\circ}\,\mathsf{C}$
$\Gamma, B \vdash A$	$\mathcal{S}$ contains no moves in B	$\dfrac{\overset{\mathfrak{D}_{\mathcal{S}}\Vert}{\Gamma \vdash A}}{\Gamma, B \vdash A}\,\mathsf{W}$

Tip: remember Currying $A \multimap (B \multimap C) = (A \wedge B) \multimap C$

Theorem (Full Completeness)

Every WIS *on* $[\![F]\!]$ *is the image of a* LI*-proof of* F*.*

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Theorem (Compositionality)

WIS*s compose nicely (well-defined, uniqueness, associativity, ...):*

$$\mathcal{S}_1 \ \textit{for}\ \Gamma \vdash A \quad \textit{and} \quad \mathcal{S}_2 \ \textit{for}\ A \vdash B \implies \mathcal{S}_1 * \mathcal{S}_2 = \mathcal{S} \ \textit{for}\ \Gamma \vdash B$$

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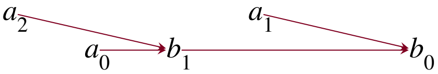
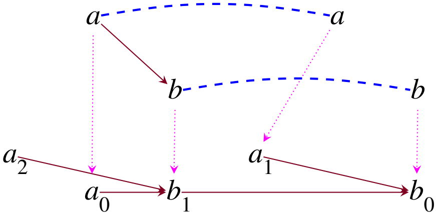
WISs are a denotational semantics for LI!

Combinatorial Proofs
for
Intuitionistic Logic



$$MAX(\mathcal{S}) = \left\{ \begin{array}{c} b_0^\circ b_1^\bullet a_0^\circ a_1^\bullet \\ b_0^\circ b_1^\bullet a_2^\circ a_1^\bullet \end{array} \right\}$$



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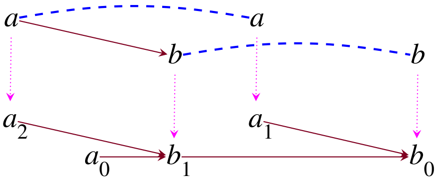

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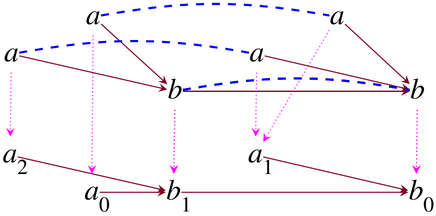

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Game Semantics
for
Constructive Modal Logic

Crash course on Constructive Modal Logic CK

$$A, B ::= 1 \mid a \mid A \supset B \mid A \wedge B \mid \Box A \mid \Diamond A$$

Crash course on Constructive Modal Logic CK

$$A, B ::= 1 \mid a \mid A \supset B \mid A \wedge B \mid \Box A \mid \Diamond A$$

Intuitionistic propositional logic (LI)

$+$

Nec rule:= if F is provable, then $\Box F$ is provable

$+$

$\mathsf{k}_1 : \Box(A \supset B) \supset (\Box A \supset \Box B)$ $\mathsf{k}_2 : \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$

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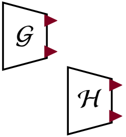
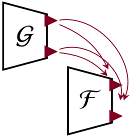
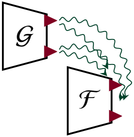
$$\frac{}{a \vdash a}\,\mathsf{AX} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,\supset^{\mathsf{R}} \quad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\,\supset^{\mathsf{L}} \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge^{\mathsf{R}} \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge^{\mathsf{L}}$$

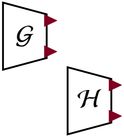
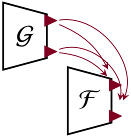
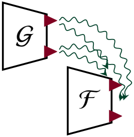
$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathsf{C} \quad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathsf{W} \quad \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\mathsf{K}^{\Box} \quad \frac{A, \Gamma \vdash B}{\Diamond A, \Box\Gamma \vdash \Diamond B}\,\mathsf{K}^{\Diamond}$$

$$
\begin{array}{lcl}
[\![a]\!] = a & [\![A \wedge B]\!] = [\![A]\!] + [\![B]\!] & [\![A \supset B]\!] = [\![A]\!] \mathbin{-\!\triangleright} [\![B]\!] \\
[\![\Box A]\!] = \Box \mathbin{\sim\!\triangleright} [\![A]\!] & [\![\Diamond A]\!] = \Diamond \mathbin{\sim\!\triangleright} [\![A]\!] &
\end{array}
$$

$\mathcal{G} + \mathcal{H}$	$\mathcal{G} \mathbin{-\!\triangleright} \mathcal{F}$	$\mathcal{G} \mathbin{\sim\!\triangleright} \mathcal{F}$	$\mathcal{G} \mathbin{-\!\triangleright} \emptyset = \Box \mathbin{\sim\!\triangleright} \emptyset$	$\Diamond \mathbin{\sim\!\triangleright} \emptyset$
			$\emptyset$	$\Diamond$

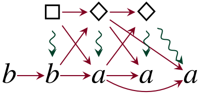
$$[\![a]\!] = a \qquad [\![A \wedge B]\!] = [\![A]\!] + [\![B]\!] \qquad [\![A \supset B]\!] = [\![A]\!] \multimap\!\triangleright [\![B]\!]$$

$$[\![\Box A]\!] = \Box \sim\!\triangleright [\![A]\!] \qquad [\![\Diamond A]\!] = \Diamond \sim\!\triangleright [\![A]\!]$$

$\mathcal{G} + \mathcal{H}$	$\mathcal{G} \multimap\!\triangleright \mathcal{F}$	$\mathcal{G} \sim\!\triangleright \mathcal{F}$	$\mathcal{G} \multimap\!\triangleright \emptyset = \Box \sim\!\triangleright \emptyset$	$\Diamond \sim\!\triangleright \emptyset$
			$\emptyset$	$\Diamond$

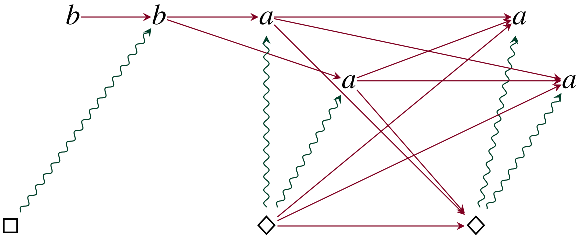
Examples:

$$[\![(\Box(b \supset b) \supset \Diamond a) \supset \Diamond(a \wedge a)]\!] =$$



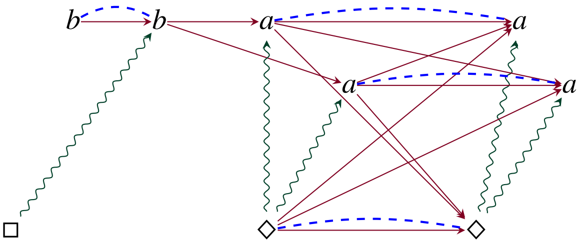
Combinatorial Proofs for CK:

- Arenas for modal formulas



Combinatorial Proofs for CK:

- Arenas for modal formulas
- Linear proofs = arenas + specific vertices partitions



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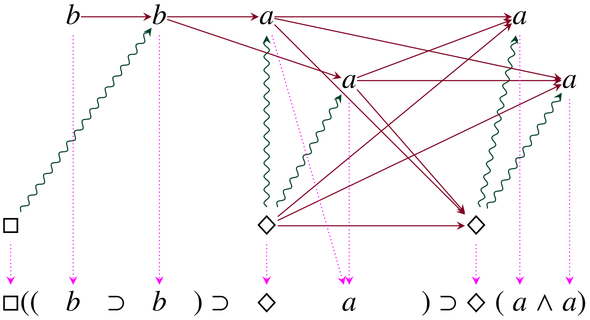

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- Arenas for modal formulas
- Linear proofs = arenas + specific vertices partitions
- Specific morphisms = deep-WC derivations
- We can factorize CK proofs

$$\Box((\ b \ \supset \ b\ ) \supset \ \Diamond \ (a \wedge a) \ ) \supset \Diamond \,(\, a \wedge a)$$

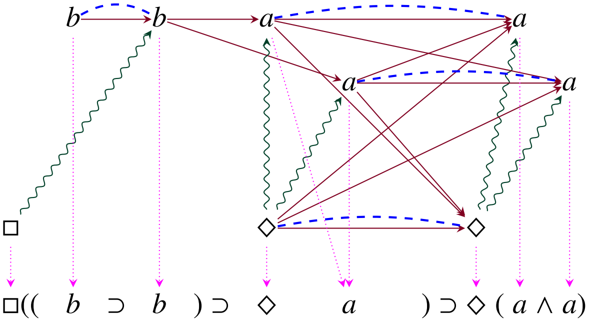
$\Vert$ IMLL-X$^{\circ}$

$\Vert$ LI$^{\circ}_{\downarrow}$

$$\Box((\ b \ \supset \ b\ ) \supset \ \Diamond \ a \ ) \supset \Diamond \,(\, a \wedge a)$$

Combinatorial Proofs for CK:

- Arenas for modal formulas
- Linear proofs = arenas + specific vertices partitions
- Specific morphisms = deep-WC derivations
- We can factorize CK proofs
- We have combinatorial proofs for CK!

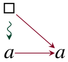


Back to games...

How to play:

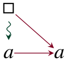
- ∘ starts on a root
- any non initial move is *justified* by a previous move
- ∘ is *shortsighted*: ∘-moves are justified by the previous •-move
- each •-move must ”reply” the previous ∘-move (same label)

Here I should have no chances to win

Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$FAIL$ $\supset^{R} \dfrac{\Box a \vdash a}{\vdash \Box a \supset a}$	$\Box a \supset a$ $\mathcal{S} = \{a^{\circ} \quad a^{\bullet}\}$

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\ \mathsf{K}^{\Box} \qquad \frac{A, \Gamma \vdash B}{\Diamond A, \Box\Gamma \vdash \Diamond B}\ \mathsf{K}^{\Diamond}$$

Here I should have no chances to win

Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$\dfrac{\dfrac{FAIL}{\Box a \vdash a}}{\vdash \Box a \supset a}\supset^{R}$	$\Box a \supset a$ $\mathcal{S} = \{a^\circ \quad \overset{\Box^\bullet}{a^\bullet}\}$

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\mathsf{K}^{\Box} \qquad \frac{A, \Gamma \vdash B}{\Diamond A, \Box\Gamma \vdash \Diamond B}\,\mathsf{K}^{\Diamond}$$

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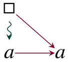
Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$\dfrac{\dfrac{FAIL}{\Box a \vdash a}}{\vdash \Box a \supset a} \supset^{R}$	$\Box a \supset a$ $\mathcal{S} = \{\overset{\epsilon}{a^{\circ}} \quad \overset{\Box^{\bullet}}{a^{\bullet}}\}$

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\mathsf{K}^{\Box} \qquad \frac{A, \Gamma \vdash B}{\Diamond A, \Box\Gamma \vdash \Diamond B}\,\mathsf{K}^{\Diamond}$$

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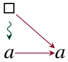
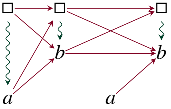
$\Box a \supset a$		$\dfrac{\dfrac{FAIL}{\Box a \vdash a}}{\vdash \Box a \supset a}\supset^{R}$	$\Box a \supset a$ $\mathcal{S} = \{\overset{\epsilon}{a^\circ} \quad \overset{\Box^\bullet}{a^\bullet}\}$
$(\Box a \supset \Box b) \supset \Box(a \supset b)$		$\dfrac{\dfrac{\dfrac{FAIL}{\vdash \Box a} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\text{AX}}{b, a \vdash b}\text{W}}{b \vdash a \multimap b}\supset^{R}}{\Box b \vdash \Box(a \supset b)}\text{K}^{\Box}}{\Box a \supset \Box b \vdash \Box(a \supset b)}\supset^{L}}{\vdash (\Box a \supset \Box b) \supset \Box(a \supset b)}\supset^{R}$	$(\Box a \supset \Box b) \supset \Box(a \supset b)$ $\mathcal{S} = \{\overset{\Box^\circ}{b^\circ} \quad \overset{\Box^\bullet}{b^\bullet} \quad \overset{\Box^\circ}{a^\circ} \quad \overset{\Box^\bullet}{a^\bullet}\}$

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\text{K}^{\Box} \qquad \frac{A, \Gamma \vdash B}{\Diamond A, \Box\Gamma \vdash \Diamond B}\text{K}^{\Diamond}$$

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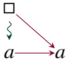
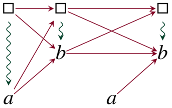
Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$\dfrac{\dfrac{FAIL}{\Box a \vdash a}}{\vdash \Box a \supset a}\,\supset^R$	$\color{red}{\Box} a \supset a$ $\mathcal{S} = \{\overset{\color{red}{\epsilon}}{a^\circ} \quad \overset{\color{red}{\Box^\bullet}}{a^\bullet}\}$
$(\Box a \supset \Box b) \supset \Box(a \supset b)$		$\dfrac{\dfrac{\dfrac{FAIL}{\vdash \Box a} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\,AX}{b, a \vdash b}\,W}{b \vdash a \multimap b}\,\supset^R}{\Box b \vdash \Box(a \supset b)}\,K^\Box}{\Box a \supset \Box b \vdash \Box(a \supset b)}\,\supset^L}{\vdash (\Box a \supset \Box b) \supset \Box(a \supset b)}\,\supset^R$	$(\color{red}{\Box} a \supset \Box b) \supset \color{blue}{\Box}(a \supset b)$ $\mathcal{S} = \{ \overset{\color{blue}{\Box^\circ}}{b^\circ} \quad \overset{\Box^\bullet}{b^\bullet} \quad \overset{\color{red}{\Box^\circ}}{a^\circ} \quad \overset{\Box^\bullet}{a^\bullet} \}$

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,K^\Box \qquad \frac{A, \Gamma \vdash B}{\Diamond A, \Box\Gamma \vdash \Diamond B}\,K^\Diamond$$

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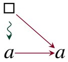
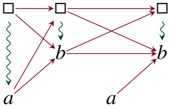
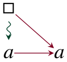
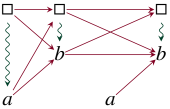
Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$\dfrac{\dfrac{FAIL}{\Box a \vdash a}}{\vdash \Box a \supset a}\supset^R$	$\Box a \supset a$ $\mathcal{S} = \{\overset{\epsilon}{a^\circ} \quad \overset{\Box^\bullet}{a^\bullet}\}$
$(\Box a \supset \Box b) \supset \Box(a \supset b)$		$\dfrac{\dfrac{\dfrac{FAIL}{\vdash \Box a} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}AX}{b, a \vdash b}W}{b \vdash a \multimap b}\supset^R}{\Box b \vdash \Box(a \supset b)}K^\Box}{\Box a \supset \Box b \vdash \Box(a \supset b)}\supset^L}{\vdash (\Box a \supset \Box b) \supset \Box(a \supset b)}\supset^R$	$(\Box a \supset \Box b) \supset \Box(a \supset b)$ $\mathcal{S} = \{\overset{\Box^\circ}{b^\circ} \quad \overset{\Box^\bullet}{b^\bullet} \quad \overset{\Box^\circ}{a^\circ} \quad \overset{\Box^\bullet}{a^\bullet}\}$

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}K^\Box \qquad \frac{A, \Gamma \vdash B}{\Diamond A, \Box\Gamma \vdash \Diamond B}K^\Diamond$$

Here I should have no chances to win

Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$\dfrac{\dfrac{FAIL}{\Box a \vdash a}}{\vdash \Box a \supset a}\supset^{R}$	$\Box a \supset a$ $\mathcal{S} = \{\overset{\epsilon}{a^{\circ}} \quad \overset{\Box^{\bullet}}{a^{\bullet}}\}$
$(\Box a \supset \Box b) \supset \Box(a \supset b)$		$\dfrac{\dfrac{\dfrac{FAIL}{\vdash \Box a} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\text{AX}}{b, a \vdash b}\text{W}}{b \vdash a \multimap b}\supset^{R}}{\Box b \vdash \Box(a \supset b)}\text{K}^{\Box}}{\Box a \supset \Box b \vdash \Box(a \supset b)}\supset^{L}}{\vdash (\Box a \supset \Box b) \supset \Box(a \supset b)}\supset^{R}$	$(\Box a \supset \Box b) \supset \Box(a \supset b)$ $\mathcal{S} = \{\ \overset{\Box^{\circ}}{b^{\circ}} \quad \overset{\Box^{\bullet}}{b^{\bullet}} \quad \overset{\Box^{\circ}}{a^{\circ}} \quad \overset{\Box^{\bullet}}{a^{\bullet}}\}$

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\text{K}^{\Box} \qquad \frac{A, \Gamma \vdash B}{\Diamond A, \Box\Gamma \vdash \Diamond B}\,\text{K}^{\Diamond}$$

Theorem (Full Completeness)

Every CK-WIS *on* $[\![F]\!]$ *is the image of a proof of* F*.*

Additional conditions on views:

1. no moves on $\Box$;
2. each $\bullet$-move is at the same “height” of the previous $\circ$-move;
3. each $\sim$-class contains a unique $\circ$-vertex;
4. each $\sim$-class contains a (unique) $\Diamond^{\circ}$ iff it contains a unique $\Diamond^{\bullet}$.

Compositionality

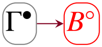
Composition = Interaction + Hide

$$\dfrac{\dfrac{\Vert \mathfrak{D}_1}{\Gamma \vdash B} \quad \dfrac{\Vert \mathfrak{D}_2}{\Delta, B \vdash C}}{\Gamma, \Delta \vdash C}\ \text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Vert \mathfrak{D}_1}{\Gamma \vdash B} \quad \dfrac{\Vert \mathfrak{D}_2}{\Delta, B \vdash C}}{\Gamma, \Delta, B \supset B \vdash C}\ \supset^{\mathsf{L}}}{\Gamma, \Delta \vdash C}\ \text{hide}$$

Composition = Interaction + Hide

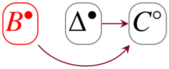
$$\dfrac{\overset{\mathfrak{D}_1}{\Gamma \vdash B} \qquad \overset{\mathfrak{D}_2}{\Delta, B \vdash C}}{\Gamma, \Delta \vdash C}\ \text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\mathfrak{D}_1}{\Gamma \vdash B} \qquad \overset{\mathfrak{D}_2}{\Delta, B \vdash C}}{\Gamma, \Delta, B \supset B \vdash C}\ \supset^{\mathsf{L}}}{\Gamma, \Delta \vdash C}\ \text{hide}$$

$[\![\mathfrak{D}_1]\!]$ for



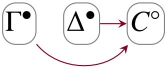
$*$

$[\![\mathfrak{D}_2]\!]$ for



$\rightsquigarrow$

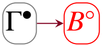
$[\![\mathfrak{D}_1]\!] *_B [\![\mathfrak{D}_2]\!]$ for



Composition = Interaction + Hide

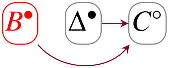
$$\dfrac{\dfrac{\mathfrak{D}_1}{\Gamma \vdash \color{red}{B}} \quad \dfrac{\mathfrak{D}_2}{\Delta, \color{red}{B} \vdash C}}{\Gamma, \Delta \vdash C}\ \text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\mathfrak{D}_1}{\Gamma \vdash \color{red}{B}} \quad \dfrac{\mathfrak{D}_2}{\Delta, \color{red}{B} \vdash C}}{\Gamma, \Delta, \color{red}{B \supset B} \vdash C}\ \supset^{\mathsf{L}}}{\Gamma, \Delta \vdash C}\ \text{hide}$$

$[\![\mathfrak{D}_1]\!]$ for



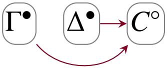
$*$

$[\![\mathfrak{D}_2]\!]$ for

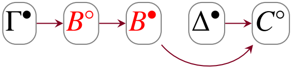


$\rightsquigarrow$

$[\![\mathfrak{D}_1]\!] *_B [\![\mathfrak{D}_2]\!]$ for



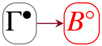
Our new approach



Composition = Interaction + Hide

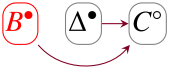
$$\dfrac{\begin{array}{c}\Vert \mathfrak{D}_1 \\ \Gamma \vdash B\end{array} \quad \begin{array}{c}\Vert \mathfrak{D}_2 \\ \Delta, B \vdash C\end{array}}{\Gamma, \Delta \vdash C}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\Vert \mathfrak{D}_1 \\ \Gamma \vdash B\end{array} \quad \begin{array}{c}\Vert \mathfrak{D}_2 \\ \Delta, B \vdash C\end{array}}{\Gamma, \Delta, B \supset B \vdash C}\,\supset^{\mathsf{L}}}{\Gamma, \Delta \vdash C}\,\text{hide}$$

$[\![\mathfrak{D}_1]\!]$ for



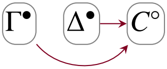
$*$

$[\![\mathfrak{D}_2]\!]$ for

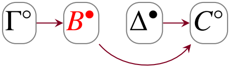


$\rightsquigarrow$

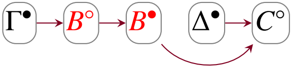
$[\![\mathfrak{D}_1]\!] *_B [\![\mathfrak{D}_2]\!]$ for



Literature approach



Our new approach

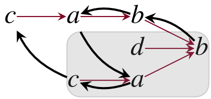


Interaction

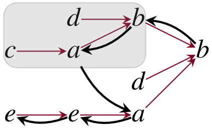
$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

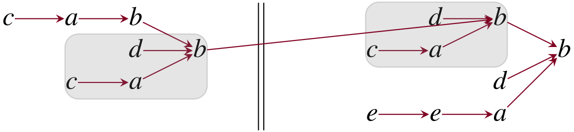


$\tau = bbaacc$



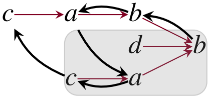
$\rho_1 = bbaaee$

$\tau \overset{B}{\circ} \rho_1 = b$

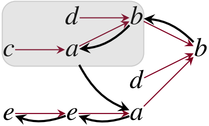


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ ‖ $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

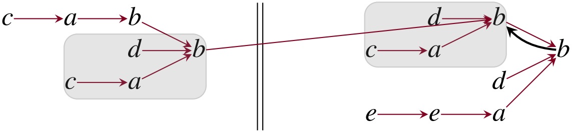


$\tau = bbaacc$



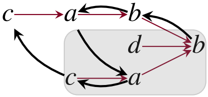
$\rho_1 = bbaaee$

$\tau \overset{B}{◐} \rho_1 = bb$

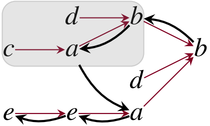


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

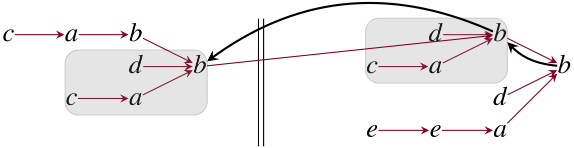


$\tau = bbaacc$



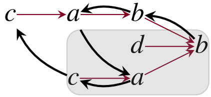
$\rho_1 = bbaaee$

$\tau \overset{B}{\bullet} \rho_1 = bbb$

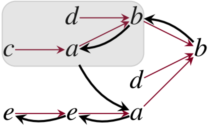


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

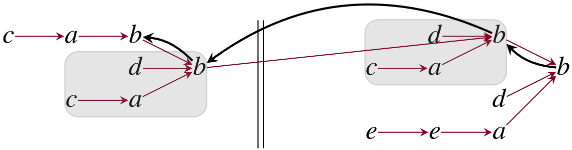


$\tau = bbaacc$



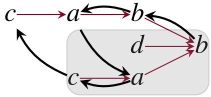
$\rho_1 = bbaaee$

$\tau \overset{B}{\circledcirc} \rho_1 = bbbb$

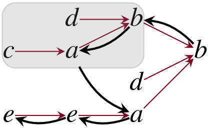


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

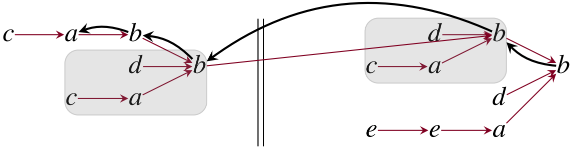


$\tau = bbaacc$



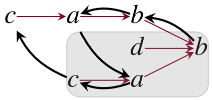
$\rho_1 = bbaaee$

$\tau \overset{B}{◐} \rho_1 = bbbba$

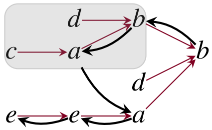


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ ‖ $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

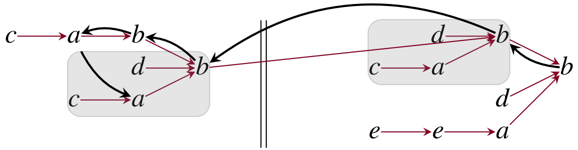


$\tau = bbaacc$



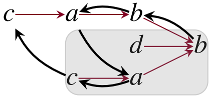
$\rho_1 = bbaaee$

$\tau \overset{B}{\bullet} \rho_1 = bbbbaa$

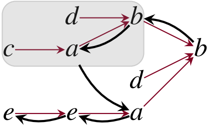


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

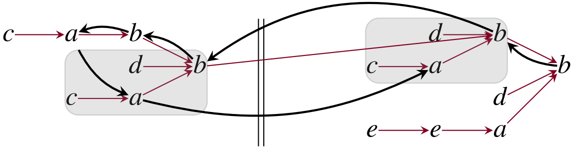


$\tau = bbaacc$



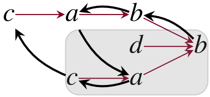
$\rho_1 = bbaaee$

$\tau \overset{B}{\LEFTcircle} \rho_1 = bbbbaaa$



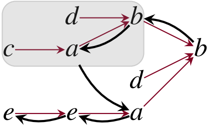
Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$



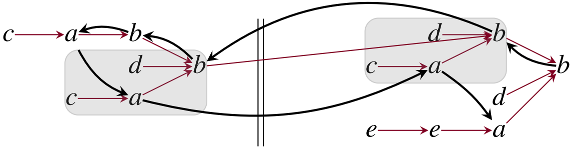
$\tau = bbaacc$

$[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$



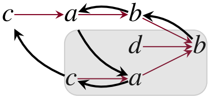
$\rho_1 = bbaaee$

$\tau \overset{B}{\circ} \rho_1 = bbbbaaaa$

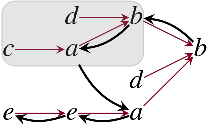


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ ‖ $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

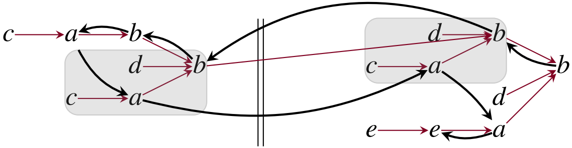


$\tau = bbaacc$



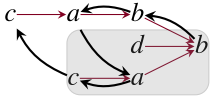
$\rho_1 = bbaaee$

$\tau \overset{B}{\bullet} \rho_1 = bbbbaaaae$

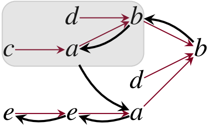


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

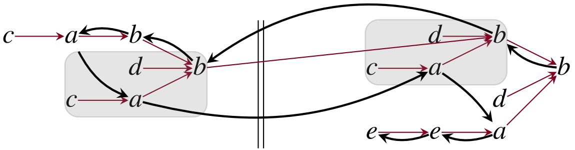


$\tau = bbaacc$



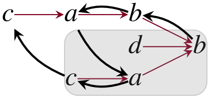
$\rho_1 = bbaaee$

$\tau \overset{B}{\circ} \rho_1 = bbbbaaaaee$

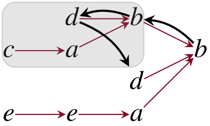


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

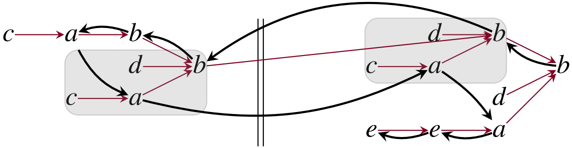


$\tau = bbaacc$



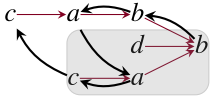
$\rho_2 = bbdd$

$\tau \overset{B}{◐} \rho_2 = b$

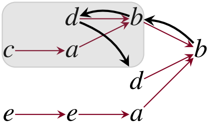


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

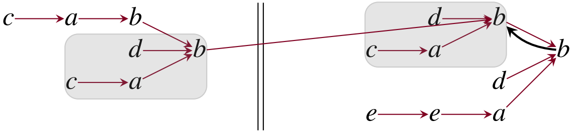


$\tau = bbaacc$



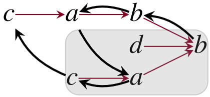
$\rho_2 = bbdd$

$\tau \overset{B}{\circ} \rho_2 = bb$

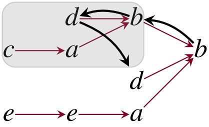


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

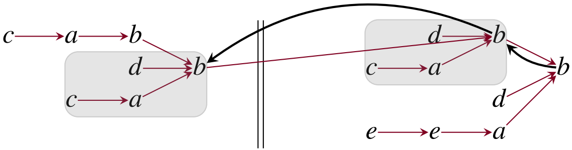


$\tau = bbaacc$



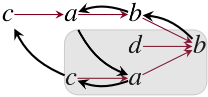
$\rho_2 = bbdd$

$\tau \overset{B}{\circ} \rho_2 = bbb$

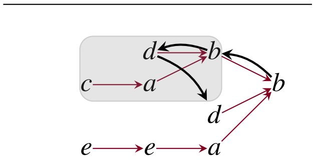


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

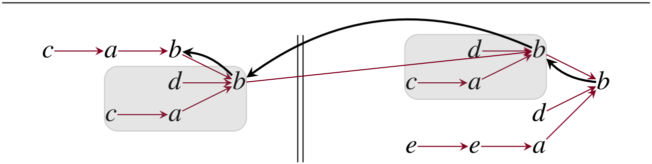


$\tau = bbaacc$



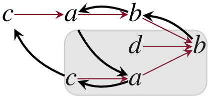
$\rho_2 = bbdd$

$\tau \overset{B}{\circ} \rho_2 = bbbb$

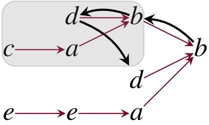


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

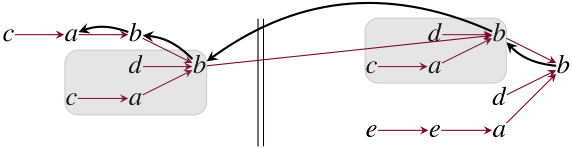


$\tau = bbaacc$



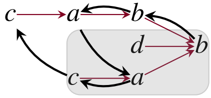
$\rho_2 = bbdd$

$\tau \overset{B}{\circ} \rho_2 = bbbba$

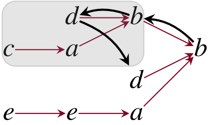


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ ‖ $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

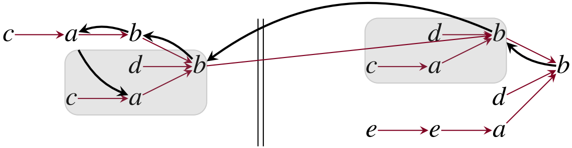


$\tau = bbaacc$



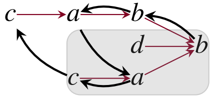
$\rho_2 = bbdd$

$\tau \overset{B}{\circ} \rho_2 = bbbbaa$

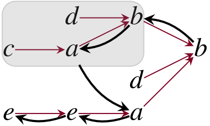


Interaction

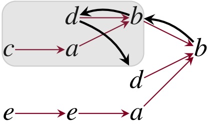
$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$



$\tau = bbaacc$

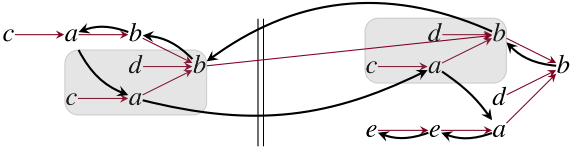


$\rho_1 = bbaaee$



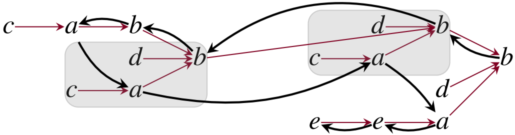
$\rho_2 = bbdd$

$$\tau \overset{B}{◐} \rho_1 = bbbbaaaaee \quad < \quad \tau \overset{B}{◐} \rho_2 = bbbbaa$$



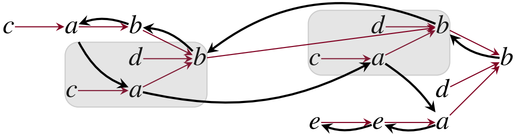
Hide

$$\tau \overset{B}{\LEFTcircle} \rho_1 = bbbbaaaaee$$



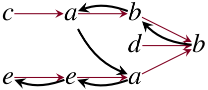
Hide

$$\tau \overset{B}{\LEFTcircle} \rho_1 = bbbbaaaaee$$



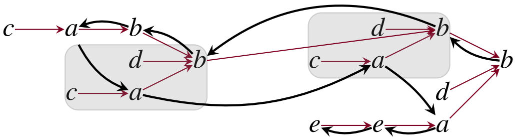
↓

$$\tau \overset{B}{*} \rho_1 = bbaaee$$



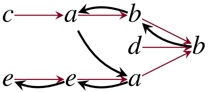
Hide

$$\tau \overset{B}{\LEFTcircle} \rho_1 = bbbbaaaaee$$



$$\downarrow$$

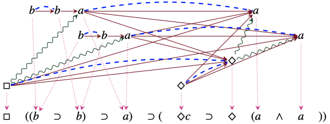
$$\tau \overset{B}{*} \rho_1 = bbaaee$$



- After hiding the $\sim$-classes "well-behave"

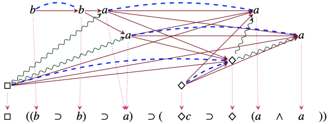
On Proof Equivalence
for
Constructive Modal Logic CK

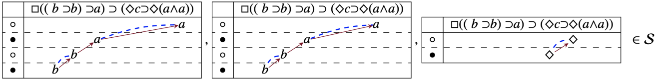
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{AX}}{\vdash b \supset b}\,\supset^{\mathsf{R}} \quad \dfrac{}{a \vdash a}\,\mathsf{AX}}{(b \supset b) \supset a \vdash a}\,\supset^{\mathsf{L}} \qquad \dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{AX}}{\vdash b \supset b}\,\supset^{\mathsf{R}} \quad \dfrac{}{a \vdash a}\,\mathsf{AX}}{(b \supset b) \supset a \vdash a}\,\supset^{\mathsf{L}}}{(b \supset b) \supset a, (b \supset b) \supset a \vdash a \wedge a}\,\wedge^{\mathsf{R}}}{(b \supset b) \supset a \vdash a \wedge a}\,\mathsf{C}}{c, (b \supset b) \supset a \vdash a \wedge a}\,\mathsf{W}}{\Diamond c, \Box((b \supset b) \supset a) \vdash \Diamond(a \wedge a)}\,\mathsf{K}^{\Diamond}}{\Box(b \supset b) \supset a) \supset (\Diamond c \supset \Diamond(a \wedge a))}\,\supset^{\mathsf{R}}$$



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{AX}}{c, b \vdash b}\,\mathsf{W}}{c \vdash b \supset b}\,\supset^{\mathsf{R}} \qquad \dfrac{\dfrac{\dfrac{}{a \vdash a}\,\mathsf{AX} \quad \dfrac{}{a \vdash a}\,\mathsf{AX}}{a, a \vdash a \wedge a}\,\wedge^{\mathsf{L}}}{a \vdash a \wedge a}\,\mathsf{C}}{c, (b \supset b) \supset a \vdash a \wedge a}\,\supset^{\mathsf{L}}}{\Diamond c, \Box((b \supset b) \supset a) \vdash \Diamond(a \wedge a)}\,\mathsf{K}^{\Diamond}}{\Box(b \supset b) \supset a) \supset (\Diamond c \supset \Diamond(a \wedge a))}\,\supset^{\mathsf{R}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{AX}}{\vdash b \supset b}\,\supset^{\mathsf{R}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a}\,\mathsf{AX} \quad \dfrac{}{a \vdash a}\,\mathsf{AX}}{a, a \vdash a \wedge a}\,\wedge^{\mathsf{L}}}{a \vdash a \wedge a}\,\mathsf{C}}{c, a \vdash a \wedge a}\,\mathsf{W}}{c, (b \supset b) \supset a \vdash a \wedge a}\,\supset^{\mathsf{L}}}{\Diamond c, \Box((b \supset b) \supset a) \vdash \Diamond(a \wedge a)}\,\mathsf{K}^{\Diamond}}{\Box(b \supset b) \supset a) \supset (\Diamond c \supset \Diamond(a \wedge a))}\,\supset^{\mathsf{R}}$$





Relation with λ-calculus

Can we extend the correspondence $\{\lambda\text{-terms}\} \leftrightarrow \{\text{WISs}\}$?

Problem:

$$t := x \mid \lambda x.t \mid (t)u \mid \text{Let } \vec{x} \text{ be } \vec{u} \text{ in } t$$

Relation with λ-calculus

Can we extend the correspondence $\{\lambda\text{-terms}\} \leftrightarrow \{\text{WISs}\}$?

Problem:

$$t := x \mid \lambda x.t \mid (t)u \mid t\left[\vec{t}/\vec{x}\right]$$

The Box-construction of the CK-lambda calculus can be read as explicit substitution

However, $x[t/y] : \Box A$ and $x : A$ are distinct terms
but $\Box A$ and A have the same strategy

Independent rules	$\equiv$
WC-interactions	$\dfrac{\dfrac{\Gamma,A,A,B,B \vdash C}{\Gamma,A,B\vdash C}\,2\times\mathsf{C}}{\Gamma,A\wedge B\vdash C}\wedge^{L} \equiv_{c} \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}\,2\times\wedge^{L}}{\Gamma,A\vdash B}\mathsf{C}$ $\dfrac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}\,2\times\mathsf{W}}{\Gamma,A\wedge B\vdash C}\wedge^{L} \equiv_{c} \dfrac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}\mathsf{C}}{\Gamma,A,A\vdash B}\mathsf{W} \equiv_{c} \Gamma,A,A\vdash B$ $\dfrac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}\mathsf{W}}{\Gamma,A\vdash B}\mathsf{C} \equiv_{c} \Gamma,A\vdash B$
Excising and Unfolding	$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta\vdash C}{B,\Delta\vdash C}\mathsf{W}}{\Gamma,\Delta,A\supset B\vdash C}\supset^{L} \equiv_{e} \dfrac{\Delta\vdash C}{\Gamma,\Delta,A\supset B\vdash C}\mathsf{W}$ $\dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}\mathsf{C}}{\Gamma,A\supset B\vdash C}\supset^{L} \equiv_{u} \dfrac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\supset^{L}}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\supset^{L}}{\Gamma,\Delta,A\supset B\vdash C}\mathsf{C}$
Structural vs K	$\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}\mathsf{W}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{K}^{\Box} \equiv_{\Box c} \dfrac{\dfrac{\Gamma\vdash A}{\Box\Gamma\vdash\Box A}\mathsf{K}^{\Box}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,B,B\vdash A}{\Gamma,B\vdash A}\mathsf{C}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{K}^{\Box} \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Box\Gamma,\Box B,\Box B\vdash\Box A}\mathsf{K}^{\Box}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{C}$ $\dfrac{\dfrac{\Gamma,B\vdash A}{\Gamma,B,C\vdash A}\mathsf{W}}{\Box\Gamma,\Diamond B,\Box C,\vdash\Box A}\mathsf{K}^{\Diamond} \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B\vdash A}{\Box\Gamma,\Diamond B\vdash\Diamond A}\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Box C\vdash\Box A}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Gamma,B,C\vdash A}\mathsf{C}}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}\mathsf{K}^{\Box} \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Box\Gamma,\Diamond B,\Box C,\Box C\Box\vdash\Diamond A}\mathsf{K}^{\Box}}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}\mathsf{C}$
Jumps	$\dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}\mathsf{W}}{\Box\Gamma,\Diamond B\vdash\Diamond A}\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}\mathsf{W} \equiv_{\mathsf{full}} \dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,C\vdash A}\mathsf{W}}{\Box\Gamma,\Diamond C\vdash\Diamond A}\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}\mathsf{W}$

$\equiv_{\mathsf{ICP}} := (\equiv \cup \equiv_{e} \cup \equiv_{c})$ $\equiv_{\lambda} := (\equiv_{\mathsf{ICP}} \cup \equiv_{u})$ $\equiv_{\mathsf{WIS}} := (\equiv_{\lambda} \cup \equiv_{\Box c})$ $\equiv_{\mathsf{full}} := (\equiv_{\mathsf{WIS}} \cup \equiv_{\Box c})$

P-Space

Future works

- Define the modal λ-calculus corresponding to our WISs
- Semantics for proof certificates using modules
- Study relation with λ-calculus with explicit substitution
- Denotational semantics (via concurrent games) for CS4

Thanks

Thanks

Questions?