Dialogical Strategies as Proof-search Strategies

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Joint works with Davide Catta

What is a proof?

A proof is...

- A strategy to win an argumentation

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- A sequence of instructions

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- The sound relations between the components of a statement

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- A strategy to win an argumentation ☜
- A sequence of instructions ☜
- The sound relations between the components of a statement

Motivations

In **dialogical logic** proofs are **winning strategies** for a two-player turn-based game.

- **Proponent** (**P**) tries to construct a proof of a formula A
- . . . by answering to the **Opponent** (**O**) objections.

An (informal) Example

Consider the formula $a \longrightarrow (b \longrightarrow a)$

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- What is the exact relation between these games and derivations?
- Can we capture proof-search strategies as **P** and **O** behavior?
- What about game semantics [denotational semantics]?

Intuitionistic Logic (minimal)

Formulas:

$$A, B ::= \quad a \quad | \quad A \to B$$

Sequent calculus $\mathrm{LJ}^{\to}$:

$$\text{ax}\,\frac{}{\Gamma, a \vdash a} \qquad \to_R \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad \to_L \frac{\Gamma, A \to B \vdash A \quad \Gamma, A \to B, B \vdash C}{\Gamma, A \to B \vdash C}$$

Proposition

$A_1, \ldots, A_n \vdash C$ is derivable in $\mathrm{LJ}^{\to}$ iff $A_1 \to (\cdots \to (A_n \to C) \cdots)$ is valid.

A lot of non-determinism in proof search!

[even in in $\mathsf{LJ}^{\rightarrow}$]

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\Gamma_1, a \vdash a$}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\Gamma_2, b \vdash b$}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\Gamma_3, c \vdash c$}
\LeftLabel{$\rightarrow_L$}
\BinaryInfC{$a \rightarrow b, {\color{red} b \rightarrow c}, b \vdash c$}
\LeftLabel{$\rightarrow_L$}
\BinaryInfC{${\color{blue} a \rightarrow b}, b \rightarrow c, a \vdash c$}
\LeftLabel{$3\times\rightarrow_R$}
\doubleLine
\UnaryInfC{$\vdash (a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c))$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\Delta_1, a \vdash a$}
\AxiomC{}
\LeftLabel{ax}
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{\scriptstyle 3\times\rightarrow_R}\dfrac{{\scriptstyle \rightarrow_L}\dfrac{{\scriptstyle ax}\dfrac{}{\Gamma_1, a \vdash a} \qquad {\scriptstyle \rightarrow_L}\dfrac{{\scriptstyle ax}\dfrac{}{\Gamma_2, b \vdash b} \quad {\scriptstyle ax}\dfrac{}{\Gamma_3, c \vdash c}}{a \rightarrow b, \textcolor{red}{b \rightarrow c}, b \vdash c}}{\textcolor{blue}{a \rightarrow b}, b \rightarrow c, a \vdash c}}{\vdash (a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c))}
\quad \simeq \quad
{\scriptstyle 3\times\rightarrow_R}\dfrac{{\scriptstyle \rightarrow_L}\dfrac{{\scriptstyle \rightarrow_L}\dfrac{{\scriptstyle ax}\dfrac{}{\Delta_1, a \vdash a} \quad {\scriptstyle ax}\dfrac{}{\Delta_2, b \vdash b}}{\textcolor{blue}{a \rightarrow b}, b \rightarrow c, a \vdash b} \qquad {\scriptstyle ax}\dfrac{}{\Delta_3, c \vdash c}}{a \rightarrow b, \textcolor{red}{b \rightarrow c}, a \vdash c}}{\vdash (a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c))}
$$

Dialogical Logic

Move: $\underbrace{\langle ?, \alpha \rangle}_{\textbf{Attack}}$ $\underbrace{\langle !, F \rangle}_{\textbf{Defense}}$ with $\alpha \in \mathcal{F} \cup \{\bullet\}$ and $F \in \mathcal{F}$

Play: sequence $\rho \coloneqq \rho_1, \ldots, \rho_n$ of moves
[each ρ_{2k} is a **P**-move, each ρ_{2k+1} is an **O**-move]

Justification: map ϕ such that $\phi(\rho_i) = \rho_j$ with $j < i$:

- $\rho_i = \langle ?, \bullet \rangle$ and $\phi(\rho_i) = \langle \star, a \rangle$;
- $\rho_i = \langle ?, A \rangle$ and $\phi(\rho_i) = \langle \star, A \to B \rangle$;
- $\rho_i = \langle !, a \rangle$ and $\phi(\rho_i) = \langle ?, \bullet \rangle$;
- $\rho_i = \langle !, B \rangle$ and $\phi(\rho_i) = \langle ?, A \rangle$;

Dialogical Play = play + justification s.t.

- $\rho_1 = \langle !, A \rangle$;
- ρ_{2k} is justified by ρ_{2k-1};
- $\rho_{2k+1} = \langle !, B \rangle$ is justified by the latest unanswered **O**-attack;
- if $\rho_i = \langle ?, \bullet \rangle$, then $i = 2k$ and $\rho_{i-1} = \langle ?, a \rangle$;
- if $\rho_{2k} = \langle !, a \rangle$, then $\rho_{2j+1} = \langle \star, a \rangle$ for a $j < k$.

P : I affirm that $a \rightarrow (b \rightarrow a)$ holds

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P : $\langle !, a \rightarrow (b \rightarrow a) \rangle$

O : $\langle ?, a \rangle$

P : $\langle !, b \rightarrow a \rangle$

O : $\langle ?, b \rangle$

P : $\langle !, a \rangle$

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P : $\langle !, a \to (b \to a) \rangle$

O : $\langle ?, a \rangle$

P : $\langle !, b \to a \rangle$

O : $\langle ?, b \rangle$

P : $\langle !, a \rangle$

$$
\begin{aligned}
\rho_i = \langle ?, \bullet \rangle &\implies \phi(\rho_i) = \langle \star, a \rangle; \\
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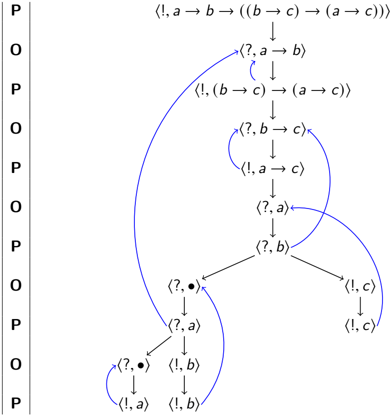
\end{aligned}
$$

Winning condition for P: ρ is finite with last (**P**-)move $\rho_{2k+1} = \langle !, a \rangle$.

Winning Strategy for P: finite tree such that

- each branch is a play for $A \in \mathcal{F}$ won by **P**;
- each **O**-move has exactly one child;
- each **P**-move has a children for each possible continuation.



(1)

Theorem (Felscher (1985), Herbelin (1995), Fermüller (2003))

There is a winning strategy for F iff F is valid.

$$
\dfrac{\dfrac{\dfrac{}{\vdash a, b \vdash a}\,\text{ax}}{\vdash a \vdash b \rightarrow a}\,\rightarrow_R}{\vdash a \rightarrow (b \rightarrow a)}\,\rightarrow_R
\quad \leftrightsquigarrow
$$

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Special Strategies

Lorenzen-Felscher:
In each play, if $\langle \star, a \rangle$ is a **P**-move, then there is a previous **O**-move $\langle \star, a \rangle$;

Stubborn:
In each play

- if $\rho \sqsupseteq \rho' \cdot \langle !, A \to B \rangle^{\mathbf{O}}$, then $\rho \sqsupseteq \rho' \cdot \langle !, A \to B \rangle^{\mathbf{O}} \cdot \langle ?, A \rangle^{\mathbf{P}}$;
- if $\rho \sqsupseteq \rho' \cdot \langle !, a \rangle^{\mathbf{O}}$, then $\rho \sqsupseteq \rho_1 \cdots \rho_{2j+1} \cdots \rho_{2k+1} \cdot \langle !, a \rangle^{\mathbf{O}} \cdot \langle ?, a \rangle^{\mathbf{P}}$;

Correspondence between strategies and derivations

- **Strategic** derivation:

$$
{\scriptstyle \to_L}\dfrac{\overset{\pi_1 \,\Vert}{\Gamma' \vdash A \to B} \qquad \overset{\pi_2 \,\Vert}{\Gamma', C \vdash D}}{\Gamma, (A \to B) \to C \vdash D}
\quad\Longrightarrow\quad
{\scriptstyle \to_L}\dfrac{{\scriptstyle \to_R}\dfrac{\overset{\pi_1' \,\Vert}{\Gamma', A \vdash B}}{\Gamma' \vdash A \to B} \qquad \overset{\pi_2 \,\Vert}{\Gamma', C \vdash D}}{\Gamma, (A \to B) \to C \vdash D}
$$

- **LF**-derivation:

$$
{\scriptstyle \to_L}\dfrac{\overset{\pi_1 \,\Vert}{\Gamma' \vdash A} \qquad \overset{\pi_2 \,\Vert}{\Gamma', B \vdash C}}{\Gamma, A \to B \vdash C}
\quad\Longrightarrow\quad
{\scriptstyle \to_L}\dfrac{{\scriptstyle \to_R}\dfrac{\vdots}{\Gamma' \vdash A} \qquad \overset{\pi_2 \,\Vert}{\Gamma', B \vdash C}}{\Gamma, A \to B \vdash C}
\quad \text{or} \quad
{\scriptstyle \to_L}\dfrac{{\scriptstyle \mathrm{ax}}\dfrac{}{\Gamma' \vdash {\color{red}a}} \qquad \overset{\pi_2 \,\Vert}{\Gamma', B \vdash C}}{\Gamma, {\color{red}a} \to B \vdash C}
$$

- **ST**-derivation:

$$
{\scriptstyle \to_L}\dfrac{\overset{\pi_1 \,\Vert}{\Gamma' \vdash A} \qquad \overset{\pi_2 \,\Vert}{\Gamma', B \vdash C}}{\Gamma, A \to B \vdash C}
\quad\Longrightarrow\quad
{\scriptstyle \to_L}\dfrac{\overset{\pi_1 \,\Vert}{\Gamma' \vdash A} \qquad {\scriptstyle \to_L}\dfrac{\vdots}{\Gamma', {\color{red}B} \vdash C}}{\Gamma, A \to B \vdash C}
\quad \text{or} \quad
{\scriptstyle \to_L}\dfrac{\overset{\pi_1 \,\Vert}{\Gamma' \vdash A} \qquad {\scriptstyle \mathrm{ax}}\dfrac{}{\Gamma', {\color{red}b} \vdash {\color{red}b}}}{\Gamma, A \to {\color{red}b} \vdash C}
$$

Example

$$
{\scriptstyle 3\times\to_R}\dfrac{{\scriptstyle \to_L}\dfrac{{\scriptstyle \mathrm{ax}}\dfrac{}{\Gamma, a \vdash a} \qquad {\scriptstyle \to_L}\dfrac{{\scriptstyle \mathrm{ax}}\dfrac{}{\Gamma, b \vdash b} \qquad {\scriptstyle \mathrm{ax}}\dfrac{}{\Gamma, c \vdash c}}{a \to b, {\color{red}b \to c}, b \vdash c}}{{\color{blue}a \to b}, b \to c, a \vdash c}}{\vdash (a \to b) \to ((b \to c) \to (a \to c))}
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$$

Theorem

There are bijections between:

- *Strategic derivations of A and winning strategies for A;*
- *LF-derivations of A and Lorenzen-Felscher winning strategies for A;*
- *ST-derivations of A and Stubborn winning strategies for A.*

Theorem

Strategic derivations, LF-derivations and ST-derivations are sound and complete for intuitionistic logic.

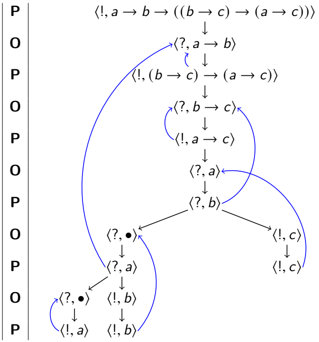
Theorem

There is a one-to-one correspondence between Stubborn winning strategies and Hyland-Ong winning innocent strategies.

Dialogical Logic	Games on Hyland-Ong arenas
a play $\sigma_1, \sigma_2, \ldots$ starts $i = 1$ odd	a play $\tau_0, \tau_1, \ldots$ starts $i = 0$ even
a play starts with a **P**-move	a play starts with a **O**-move
a move is a subformula of F plus a polarity	a move corresponds to an atom in F

Corollary

There is a one-to-one correspondence between Stubborn winning strategies and λ-terms in $\eta\beta$-normal form.

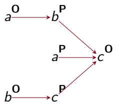


$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, a \vdash a}\,\text{ax} \quad \dfrac{}{\Gamma, b \vdash b}\,\text{ax}}{a \to b, b \to c, b \vdash c}\,{\to_L} \quad \dfrac{}{\Gamma, c \vdash c}\,\text{ax}}{a \to b, b \to c, a \vdash c}\,{\to_L}}{\vdash (a \to b) \to ((b \to c) \to (a \to c))}\,3{\times}{\to_R}$$

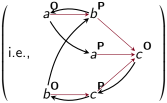
$\lambda x.\lambda y.\lambda z.yxz$

$\Sigma = \{c^{\mathbf{O}} c^{\mathbf{P}} b^{\mathbf{O}} b^{\mathbf{P}} a^{\mathbf{O}} a^{\mathbf{P}}\}$

over



i.e.,



Conclusion and Future Works

Main results:

- Correspondence between dialogical games and sequent calculi [between restriction on plays and proof search strategy]
- Correspondence between dialogical games and game semantics [between Lorenz&Lorenzen games and Hyland-Ong games]

Future Works:

- Extensions to the full propositional intuitionistic logic
- What about other logics? (modal, first-order, etc. . .)
- What about games with loops?