Sequent Systems on Undirected Graphs

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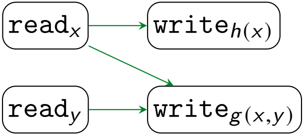
Why graphs?


a
b
c
d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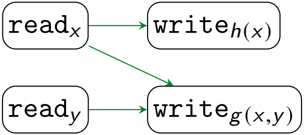
write$_{h(x)}$
read$_y$
write$_{g(x,y)}$



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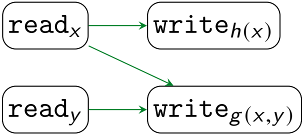

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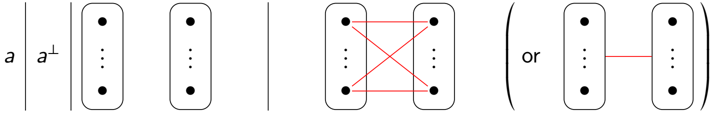
Classical Formulas and Cographs

Classical Logic Formulas

$$\phi, \psi := a \mid a^{\perp} \mid \phi \vee \psi \mid \phi \wedge \psi$$

Cographs



$$[\![\cdot]\!] : \{\text{Formulas}\} \longrightarrow \{\text{Cographs}\}$$

Note: logical negation = complementary graph (+ dual labels)

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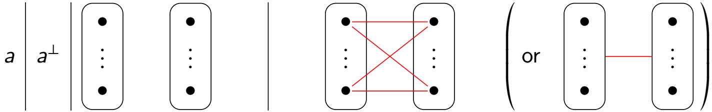

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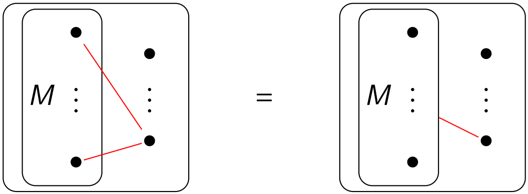
Note: logical negation = complementary graph (+ dual labels)

Tip: edge between a and b in $[\![A]\!]$ $\iff$ a and b meets in a $\wedge$ in A

From Formulas-as-Graphs to Graphs-as-Formulas

Modules

A **module** of a graph $G = H[M]$ is a set of vertices M s.t.



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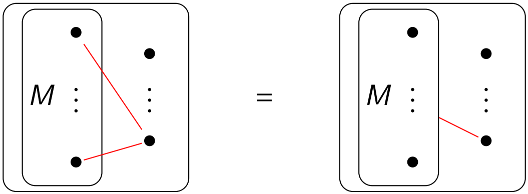

A graph G is **prime** if it has modules V_G, $\varnothing$ and $\{x\}$ for all $x \in V_G$.

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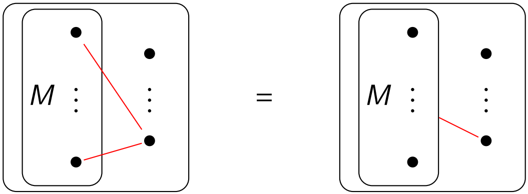

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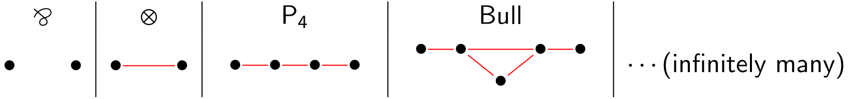

Modular decomposition

Notation
$G(\!|H_1, \ldots, H_n|\!)$ is the graph obtained by replacing vertices in G with graphs "modularly".

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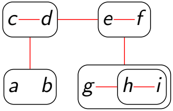

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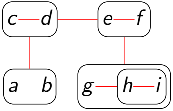

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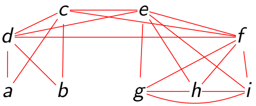


Lemma (Modular decomposition of graphs (Gallai '75))

If $G \neq \emptyset$ is a graph, then we have exactly one of the following cases:

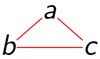
- (i) *G is a singleton graph*
- (ii) *$G = P(\!|A_1, \ldots, A_n|\!)$ for a prime graph P*

Too many denotations for the same graph

Sources of homonymity:

- Associativity of $\parr$ and $\otimes$

$$a \otimes (b \otimes c) =$$



$$= (a \otimes b) \otimes c$$

- Graphs isomorphism:

$$\mathrm{P}_4(\!|a, b, c, d|\!) = a — b — c — d = \mathrm{P}_4(\!|d, c, b, a|\!)$$

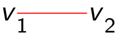
- Graphs symmetries:

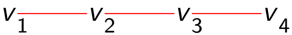
$$\underbrace{\mathrm{P}_4}_{a—b—c—d}(\!|a, b, c, d|\!) = a — b — c — d = \underbrace{\mathrm{P}_4{}'}_{c—a—d—b}(\!|b, d, a, c|\!)$$

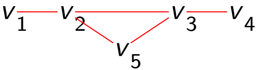
Graphical connectives (Graphs)

For each (family of symmetric) prime graphs, we fix an order on vertices:

- $\parr \llparenthesis v_1, v_2 \rrparenthesis$: 

- $\otimes \llparenthesis v_1, v_2 \rrparenthesis$: 

- $\mathrm{P}_4 \llparenthesis v_1, v_2, v_3, v_4 \rrparenthesis$: 

- $\mathrm{Bull} \llparenthesis v_1, \ldots, v_5 \rrparenthesis$: 

- ...

Formula representation of graphs

Formulas

$$\phi_1, \ldots, \phi_n \coloneqq \circ \mid a \mid a^\perp \mid \kappa_P (\!| \phi_1, \ldots, \phi_{n_P} |\!)$$

Interpretation as graphs

$$[\![\circ]\!] = \varnothing \qquad [\![a]\!] = a \qquad \left[\!\left[a^\perp\right]\!\right] = a^\perp$$

$$\left[\!\left[\kappa_P (\!| \phi_1, \ldots, \phi_n |\!)\right]\!\right] = P(\!| [\![\phi_1]\!], \ldots, [\![\phi_n]\!] |\!)$$

Previous and Related Works

Original research question:

Proof Theory treating the happens-before relation "logically"

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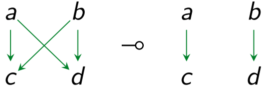

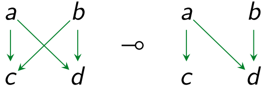

A. et al

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$$((a^\perp \otimes b^\perp) \triangleleft (c^\perp \otimes d^\perp)) \parr (a \triangleleft c) \otimes (b \triangleleft d)$$

Provable in BV

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$$\frac{((a^\perp \otimes b^\perp) \triangleleft (c^\perp \otimes d^\perp)) \parr \mathsf{N}(\!|a, b, c, d|\!)}{\text{Provable in } \mathsf{GV^{sl}}}$$

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Why deep inference?

Previous results:

- Proof systems (deep inference) operating on undirected graphs (GS), and mixed graphs (GV and GV^{sl});
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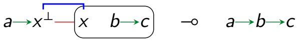
Why deep inference?

Theorem (Tiu 2006)

No possible sequent systems for BV.



$$
\text{i}\!\uparrow \frac{\circ}{(a^\perp \lhd b^\perp \lhd c^\perp) \parr (a \lhd b \lhd c)}
$$

$$
\text{asso} \frac{\left(\text{q}\!\downarrow \frac{\text{s} \frac{\left(a^\perp \lhd \left(\left(\frac{\circ}{x \parr x^\perp}\right) \otimes (b^\perp \lhd c^\perp)\right)\right)}{(a^\perp \lhd (x \parr (x^\perp \otimes (b^\perp \lhd c^\perp))))}}{(a^\perp \lhd x) \parr (x^\perp \otimes (b^\perp \lhd c^\perp))} \right) \parr (a \lhd b \lhd c)}{(a^\perp \lhd b^\perp \lhd c^\perp) \parr (a \lhd b \lhd c)}
$$

Das et al

Extension of boolean logic [Calk, Das, Waring ArXiv2020,]:

- Two enteilement mechanisms

$$\begin{array}{lllll} A \Rightarrow_{\wedge} B & \forall C_A \text{ max-Cl in } A & \exists C_B \text{ max-Cl in } B & \text{s.t.} & C_B \subseteq C_A \\ A \Rightarrow_{\vee} B & \forall S_B \text{ max-St in } B & \exists S_A \text{ max-St in } A & \text{s.t.} & S_A \subseteq C_B \end{array}$$

- Similar idea in Pratt's “Modeling concurrency with partial orders” 1986;
- Linear Inferences [Das& Rice FSCD2021&LMCS2023];
- Conservative extension of LK.

Why deep inference?

Context-free rewriting rules.

Sequent calculi operating on Graphs

Rules

$$\text{ax}\ \frac{}{\vdash a, a^{\perp}}\ a \in \mathcal{A} \qquad ⅋\ \frac{\vdash \Gamma, \phi, \psi}{\vdash \Gamma, \phi ⅋ \psi} \qquad \otimes\ \frac{\vdash \Gamma, \phi \quad \vdash \psi, \Delta}{\vdash \Gamma, \phi \otimes \psi, \Delta}$$

$$\text{d-}\kappa\ \frac{\vdash \Gamma_1, \phi_{\sigma(1)}, \psi_{\tau(1)} \qquad \cdots \qquad \vdash \Gamma_n, \phi_{\sigma(n)}, \psi_{\tau(n)}}{\vdash \Gamma_1, \ldots, \Gamma_n, \kappa(\!|\phi_1, \ldots, \phi_n|\!), \kappa^{\perp}(\!|\psi_1, \ldots \psi_n|\!)}\ \begin{cases} \sigma \in \mathfrak{S}(\kappa) \\ \tau \in \mathfrak{S}(\kappa^{\perp}) \end{cases}$$

$$\text{w}\ \frac{\vdash \Gamma}{\vdash \Gamma, \phi} \qquad \text{c}\ \frac{\vdash \Gamma, \phi, \phi}{\vdash \Gamma, \phi}$$

$$\text{mix}\ \frac{\vdash \Gamma_1 \quad \vdash \Gamma_2}{\vdash \Gamma_1, \Gamma_2} \qquad \text{wd}_{\otimes}\ \frac{\vdash \Gamma, \phi_k \quad \vdash \Delta, \kappa(\!|\phi_1, \ldots, \phi_{k-1}, \circ, \phi_{k+1}, \ldots, \phi_n|\!)}{\vdash \Gamma, \Delta, \kappa(\!|\phi_1, \ldots, \phi_n|\!)}$$

$$\text{u}_{\kappa}\ \frac{\vdash \Gamma, \chi(\!|\phi_{\sigma(1)}, \ldots, \phi_{\sigma(n)}|\!)}{\vdash \Gamma, \kappa(\!|\phi_1, \ldots, \phi_k, \circ, \phi_{k+1}, \ldots, \phi_n|\!)}\ \begin{cases} \sigma \in \mathfrak{S}(\chi) \\ [\![\kappa(\!|\phi_1, \ldots, \phi_k, \circ, \phi_{k+1}, \ldots, \phi_n|\!)]\!] = [\![\chi(\!|\phi_{\sigma(1)}, \ldots, \phi_{\sigma(n)}|\!)]\!] \neq \emptyset \end{cases}$$

$$\begin{aligned} \text{MGL} &= \{\text{ax}, ⅋, \otimes, \text{d-}P\} \\ \text{MGL}^{\circ} &= \text{MGL} \cup \{\text{mix}, \text{wd}_{\otimes}, \text{u}_{\kappa}\} \\ \text{KGL} &= \text{MGL} \cup \{\text{w}, \text{c}\} \end{aligned}$$

Main results

Theorem (Graph isomorphism)

If $[[\phi]] = [[\psi]]$ *then* $\vdash_{\mathsf{MGL}^\circ} \phi \multimap\!\circ\, \psi$. *(If* ϕ *and* ψ *unit-free, then* MGL*)*

Theorem (Cut-elimination)

The rule cut *is admissible;*

Theorem (Conservativity)

$$\mathsf{MGL} \supset \mathsf{MLL} \qquad \mathsf{MGL}^\circ \supset \mathsf{MLL}^\circ \qquad \mathsf{KGL} \supset \mathsf{LK}$$

Note: $A \star \varnothing = \star(A, \varnothing) = A$

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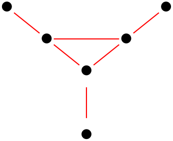
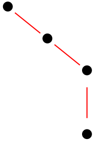
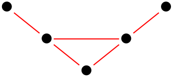
A new notion of *"sub-formula"* analyticity arises from this work:

 has sub-connective (e.g.)  and 

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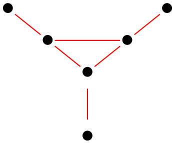
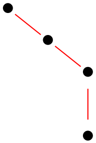
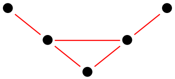
 has sub-connective (e.g.)  and 

Analytic proof: no "new connectives" occurs during proof search

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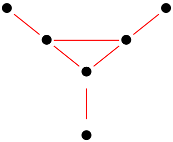

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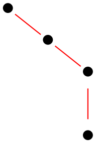

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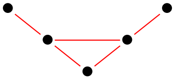


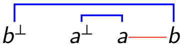
Analytic proof: no "new connectives" occurs during proof search

Theorem

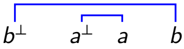
If ϕ is provable, than ϕ admits an analytic proof.

Future Works

- Mathematical structure of graphs (w.r.t. modular decomposition). Are there similar structures?
- Topological characterization (beyond Retore's criterion)

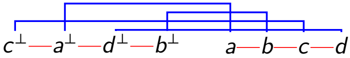


Connected+Æ-acyclic (MLL)



Æ-acyclic (MLL°)

Æ-Acyclic (but not in GS)



Æ-Acyclic (but in GS)

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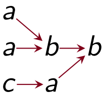
$$[[((a \wedge a) \longrightarrow b) \longrightarrow ((c \longrightarrow a) \longrightarrow b)]] =$$



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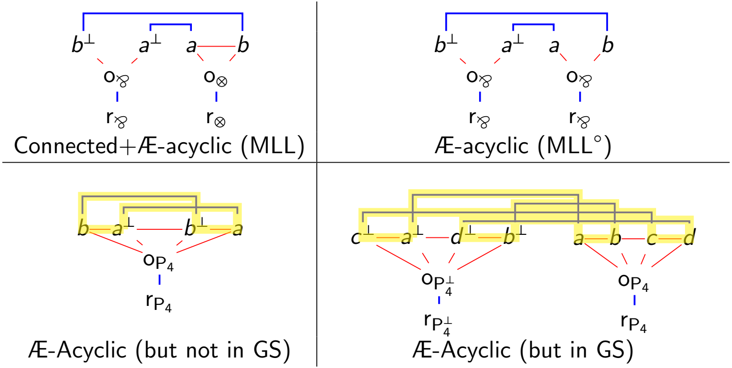

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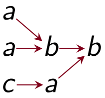
$$[\![((a \wedge a) \longrightarrow b) \longrightarrow ((c \longrightarrow a) \longrightarrow b)]\!] =$$



To sum up:

- Graphs naturally represent complex patterns of interaction;
- We can define proof systems operating on graphs!
- New graphs-as-connectives approach.

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Challenge for the ATP community

Define efficient theorem provers operating on graphs

Thank you

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Questions?
Comments?