Constructive Modal Logic: Game Semantics and Lambda-Calculi

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Based on joint works with Davide Catta[1,3], Federico Olimpieri[3], and Lutz Straßburger[1,2]

1=[Tableaux2021] 2=[AiML2022] 3=[Tableaux2023]

What is a proof?

A proof is...

- A sequence of instructions

A proof is...

- A sequence of instructions
- A strategy to win an argumentation

A proof is...

- A sequence of instructions
- A strategy to win an argumentation
- The sound relations between the components of a statement

When two proofs are the same?

- **Normalization**: $\pi_1 = \pi_2 \iff \exists\hat{\pi}$ s.t. $\pi_1 \rightsquigarrow \hat{\pi}$ and $\pi_2 \rightsquigarrow \hat{\pi}$
  - Normalization may forget information (see classical logic);
  - This approach is used to define categorical semantics and denotational semantics (including game semantics);
  - Curry-Howard correspondence: two programs are the same if they compute the same function;
- **Generality**: $\pi_1 = \pi_2 \iff [\![\pi_1]\!] = [\![\pi_2]\!]$
  - two proofs are equivalent if we can associate both a same mathematical object;
  - No normalization is involved: two programs computing a same function can still be different.

The intuitionisitc logic case

Crash course on (disjunction free) intuitionistic Logic

$$A, B ::= 1 \mid a \mid A \supset B \mid A \wedge B$$

Sequent Calulus

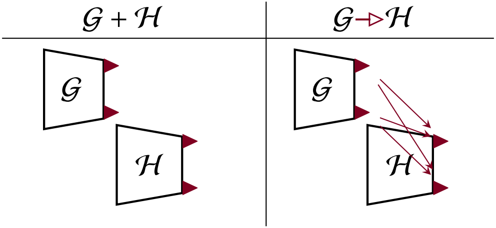
$$\frac{}{a \vdash a}\,\mathsf{AX} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,\supset^{\mathsf{R}} \quad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\,\supset^{\mathsf{L}} \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge^{\mathsf{R}} \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge^{\mathsf{L}}$$

$$\frac{}{\vdash 1}\,1 \quad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathsf{C} \quad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathsf{W}$$

Game Semantics for Intuitionistic Logic

Arenas:

$$[\![a]\!] = a \qquad [\![1]\!] = \emptyset \qquad [\![A \wedge B]\!] = [\![A]\!] + [\![B]\!] \qquad [\![A \supset B]\!] = [\![A]\!] \mathrel{-\!\!\triangleright} [\![B]\!]$$



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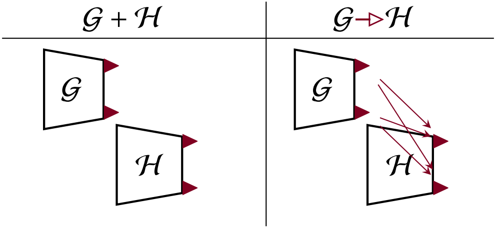


Examples:

$$[\![((b_1 \supset b_0) \supset a_1) \supset (a_2 \wedge a_0)]\!] = \begin{tikzcd} b_1 \arrow[r] & b_0 \arrow[r] & a_1 \arrow[r] \arrow[rr, bend right] & a_2 & a_0 \end{tikzcd}$$

$$[\![((a \wedge a) \supset b) \supset (a \supset b)]\!] = \begin{tikzcd} a \arrow[rr, bend right] & a \arrow[r] & b \arrow[rr, bend right] & a \arrow[r] & b \end{tikzcd}$$

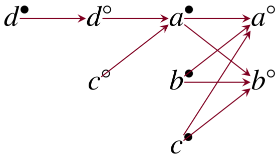
How to play:

- Two-players game (∘ and •)
- ∘ starts on a root
- each non initial move is *justified* ($\rightarrow$) by one previous move
- each •-move must “reply” to the previous ∘-move
- ∘-moves are justified by the previous •-move (∘ is *shortsighted*)
- a player wins when the other is out of moves

$$\begin{tikzcd}
d \arrow[r] & d \arrow[r] & a \arrow[r] \arrow[dr] & a \\
 & c \arrow[ur] & b \arrow[ur] \arrow[r] & b \\
 & & c \arrow[uur] \arrow[ur] &
\end{tikzcd}$$

How to play:

- Two-players game ($\circ$ and $\bullet$)
- $\circ$ starts on a root
- each non initial move is *justified* ($\rightarrow$) by one previous move
- each $\bullet$-move must “reply” to the previous $\circ$-move
- $\circ$-moves are justified by the previous $\bullet$-move ($\circ$ is *shortsighted*)
- a player wins when the other is out of moves



"A strategy to win an argument on the truthful of a statement"

- Play: sequence of moves
- Winning strategy: set of plays considering <u>every</u> possible $\circ$-move
- Innocent: each $\bullet$-move is determined by <u>one</u> previous $\circ$-move.

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

It is $\circ$'s turn

$$\begin{tikzcd}
a_2 \arrow[drr] & & & a_1 \arrow[dr] & \\
& a_0 \arrow[r] & b_1 \arrow[rr] & & b_0
\end{tikzcd}$$

$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ \\ \\ \\ \end{array} \right\}$$

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

It is $\bullet$'s turn

$$\begin{tikzcd}
a_2 \arrow[drr] & & & a_1 \arrow[dr] & \\
& a_0 \arrow[r] & b_1 \arrow[rr] & & b_0{}^{\circ}
\end{tikzcd}$$

$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^{\circ} \\ \\ \\ \\ \\ \end{array} \right\}$$

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

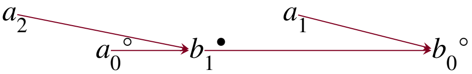
It is $\circ$'s turn

$$
\begin{tikzcd}
a_2 \arrow[drr] & & & a_1 \arrow[dr] & \\
& a_0 \arrow[r] & b_1^{\bullet} \arrow[rr] & & b_0^{\circ}
\end{tikzcd}
$$

$$
\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} \\ \\ \\ \\ \end{array} \right\}
$$

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

It is $\bullet$'s turn

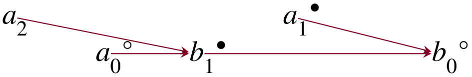


$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^\circ \\ b_0^\circ b_1^\bullet \\ b_0^\circ b_1^\bullet a_0^\circ \\ \\ \\ \\ \end{array} \right\}$$

Let's play on $[\![((a \land a) \supset b) \supset (a \supset b)]\!]$

It is $\circ$'s turn

PLAYER $\bullet$ WINS!



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It is $\circ$'s turn

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a_2 \arrow[drr] & & & a_1 \arrow[dr] & \\
& a_0 \arrow[r] & b_1 \arrow[rr] & & b_0{}^{\circ}
\end{tikzcd}$$

$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} \\ b_0^{\circ} b_1^{\bullet} a_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} a_0^{\circ} a_1^{\bullet} \\ \\ \\ \end{array} \right\}$$

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

It is $\bullet$'s turn

$$\begin{tikzcd} a_2 \arrow[drr] & & & a_1 \arrow[drr] & & \\ & a_0 \arrow[r] & b_1^{\bullet} \arrow[rrr] & & & b_0^{\circ} \end{tikzcd}$$

$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} \\ b_0^{\circ} b_1^{\bullet} a_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} a_0^{\circ} a_1^{\bullet} \\ \\ \\ \end{array} \right\}$$

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

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$$\begin{tikzcd}
a_2^{\circ} \arrow[drr] & & & a_1 \arrow[dr] & \\
& a_0 \arrow[r] & b_1^{\bullet} \arrow[rr] & & b_0^{\circ}
\end{tikzcd}$$

$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} \\ b_0^{\circ} b_1^{\bullet} a_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} a_0^{\circ} a_1^{\bullet} \\ b_0^{\circ} b_1^{\bullet} a_2^{\circ} \\ \\ \end{array} \right\}$$

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

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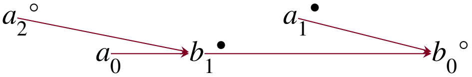


$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^\circ \\ b_0^\circ b_1^\bullet \\ b_0^\circ b_1^\bullet a_0^\circ \\ b_0^\circ b_1^\bullet a_0^\circ a_1^\bullet \\ b_0^\circ b_1^\bullet a_2^\circ \\ b_0^\circ b_1^\bullet a_2^\circ a_1^\bullet \end{array} \right\}$$

Theorem (Compositionality)

If $\mathcal{S}$ is a WIS *for* $[\![A \supset B]\!]$ *and* $\mathcal{T}$ *is a* WIS *for* $[\![B \supset C]\!]$*,*
then there is a WIS $\mathcal{S} \circ \mathcal{T}$ *for* $[\![A \supset C]\!]$*.*

Theorem (Denotational Semantics)

WISs *provide a full complete denotational semantics for intuitionistic logic.*

- *If $\mathcal{S}$ is a* WIS*, then there is π s.t. $\mathcal{S} = [\![\pi]\!]$*
- $\pi_1 \rightsquigarrow \hat{\pi} \leftsquigarrow \pi_2 \iff [\![\pi_1]\!] = [\![\pi_2]\!]$

Theorem

One-to-one correspondence between $\beta\eta$-normal λ-terms and WISs*.*

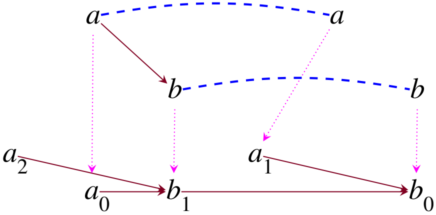
$$t := \star \mid x \mid \lambda x.t \mid (t)u \mid \langle t_1, t_2\rangle \mid \Pi_1 t \mid \Pi_2 t$$

$$(\lambda x.t)u \rightsquigarrow_\beta t\,\{u/x\} \quad \Pi_1\langle u, v, \rightsquigarrow_\beta\rangle u \quad \Pi_2\langle u, v, \rightsquigarrow_\beta\rangle v$$
$$\lambda x.t(x) \rightsquigarrow_\eta t \qquad \langle \Pi_1 u, \Pi_2 u\rangle \rightsquigarrow_\eta u$$

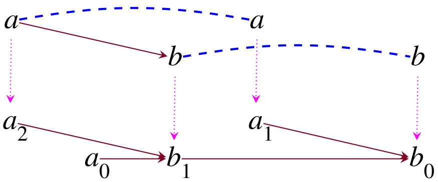
Combinatorial Proofs
for
Intuitionistic Logic

$$\begin{tikzcd}
a_2 \arrow[drr] & & & a_1 \arrow[dr] & \\
& a_0 \arrow[r] & b_1 \arrow[rr] & & b_0
\end{tikzcd}$$

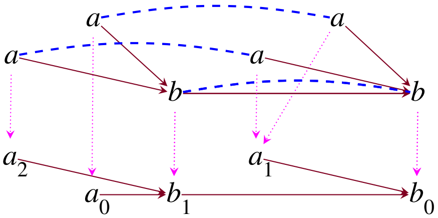
$$MAX(\mathcal{S}) = \left\{ \begin{array}{c} b_0^{\circ} b_1^{\bullet} a_0^{\circ} a_1^{\bullet} \\ b_0^{\circ} b_1^{\bullet} a_2^{\circ} a_1^{\bullet} \end{array} \right\}$$



$$MAX(\mathcal{S}) = \left\{ \begin{array}{c} b_0^{\circ} b_1^{\bullet} a_0^{\circ} a_1^{\bullet} \\ b_0^{\circ} b_1^{\bullet} a_2^{\circ} a_1^{\bullet} \end{array} \right\} \quad \leftarrow$$



$$MAX(\mathcal{S}) = \left\{ \begin{array}{c} b_0^\circ b_1^\bullet a_0^\circ a_1^\bullet \\ b_0^\circ b_1^\bullet a_2^\circ a_1^\bullet \end{array} \right\} \quad \leftarrow$$

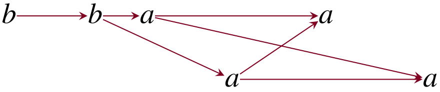


$$MAX(\mathcal{S}) = \left\{ \begin{array}{c} b_0^{\circ} b_1^{\bullet} a_0^{\circ} a_1^{\bullet} \\ b_0^{\circ} b_1^{\bullet} a_2^{\circ} a_1^{\bullet} \end{array} \right\}$$

This is an intuitionistic combinatorial proof!

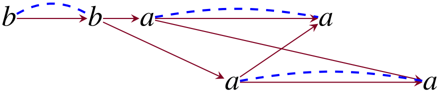
Combinatorial Proofs for LI:

- Arenas for formulas



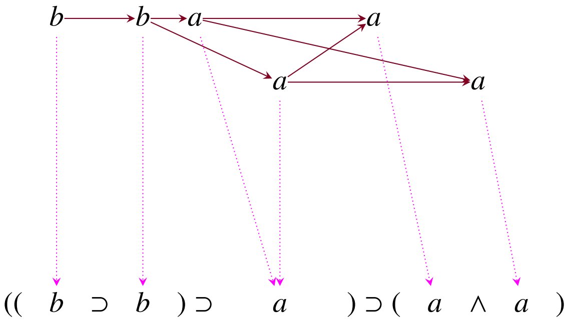
Combinatorial Proofs for LI:

- Arenas for formulas
- Linear proofs = arenas + specific vertices partitions



Combinatorial Proofs for LI:

- Arenas for formulas
- Linear proofs = arenas + specific vertices partitions
- Deep-WC derivations = specific morphisms between areans



Combinatorial Proofs for LI:

- Arenas for formulas
- Linear proofs = arenas + specific vertices partitions
- Deep-WC derivations = specific morphisms between areans
- We can factorize LI proofs

$$\overline{\|}\,\mathsf{IMLL}^{\bullet}$$

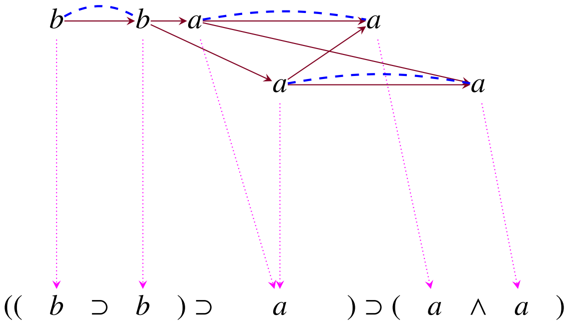
$$((\quad b \quad \supset \quad b \quad) \supset \quad (a \wedge a) \quad) \supset (\quad a \quad \wedge \quad a \quad)$$

$$\|\,\mathsf{LI}^{\bullet}_{\downarrow}$$

$$((\quad b \quad \supset \quad b \quad) \supset \quad a \quad) \supset (\quad a \quad \wedge \quad a \quad)$$

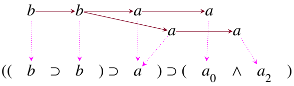
Combinatorial Proofs for LI:

- Arenas for formulas
- Linear proofs = arenas + specific vertices partitions
- Deep-WC derivations = specific morphisms between areans
- We can factorize LI proofs
- Et Voilá!

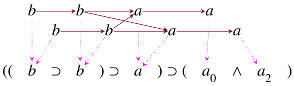


Proof equivalence in LI

Combinatorial Proofs provide a finer notion of proof equivalence w.r.t. WIS.



$\neq$



$$\dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{ax}}{\vdash b \supset b}\supset^{\mathsf{R}} \qquad \dfrac{\dfrac{\dfrac{}{a \vdash a_0}\,\mathsf{ax} \quad \dfrac{}{a \vdash a_2}\,\mathsf{ax}}{{\color{red}a, a} \vdash a_0 \wedge a_2}\wedge}{{\color{red}a} \vdash a_0 \wedge a_2}\,\mathsf{C}}{(b \supset b) \supset a \vdash a_0 \wedge a_2}\supset^{\mathsf{L}}}{\vdash ((b \supset b) \supset a) \supset (a_0 \wedge a_2)}\supset^{\mathsf{R}}$$

$\neq$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{ax}}{\vdash b \supset b}\supset^{\mathsf{R}} \quad \dfrac{}{a \vdash a_0}\,\mathsf{ax}}{(b \supset b) \supset a \vdash a_0}\supset^{\mathsf{L}} \qquad \dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{ax}}{\vdash b \supset b}\supset^{\mathsf{R}} \quad \dfrac{}{a \vdash a_2}\,\mathsf{ax}}{(b \supset b) \supset a \vdash a_2}\supset^{\mathsf{L}}}{{\color{red}(b \supset b) \supset a, (b \supset b) \supset a} \vdash a_0 \wedge a_2}\wedge}{{\color{red}(b \supset b) \supset a} \vdash a_0 \wedge a_2}\,\mathsf{C}}{\vdash ((b \supset b) \supset a) \supset (a_0 \wedge a_2)}\supset^{\mathsf{R}}$$

but

$$\mathcal{S} = \left\{ \epsilon, \begin{array}{c} a_0 \,,\, a_0 a \,,\, a_0 ab \,,\, a_0 abb \\[1em] a_2 \,,\, a_2 a \,,\, a_2 ab \,,\, a_2 abb \end{array} \right\}$$

$$\simeq$$

$$\lambda f^{(b \supset b) \supset a}.\langle f(\lambda x^a.x), f(\lambda y^a.y)\rangle$$

Combinatorial Proofs provide a finer notion of proof equivalence w.r.t. WIS.

Independent rules

$$\dfrac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$$

$$\dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1 \qquad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$$

Resource Management

$$\dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A,B\vdash C}2\times\mathsf{C}}{\Gamma,A\wedge B\vdash C}\wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}2\times\wedge^{\mathsf{L}}}{\Gamma,A\vdash B}\mathsf{C} \qquad \dfrac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}2\times\mathsf{W}}{\Gamma,A\wedge B\vdash C}\wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}\mathsf{W}$$

$$\dfrac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}\mathsf{C}}{\Gamma,A,A\vdash B}\mathsf{W} \equiv_{\mathsf{c}} \Gamma,A,A\vdash B \qquad \dfrac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}\mathsf{W}}{\Gamma,A\vdash B}\mathsf{C} \equiv_{\mathsf{c}} \Gamma,A\vdash B$$

Excising and Unfolding

$$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta\vdash C}{B,\Delta\vdash C}\mathsf{W}}{\Gamma,\Delta,A\supset B\vdash C}\supset^{\mathsf{L}} \equiv_{\mathsf{e}} \dfrac{\Delta\vdash C}{\Gamma,\Delta,A\supset B\vdash C}\mathsf{W} \qquad \dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}\mathsf{C}}{\Gamma,A\supset B\vdash C}\supset^{\mathsf{L}} \equiv_{\mathsf{u}} \dfrac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\supset^{\mathsf{L}}}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\supset^{\mathsf{L}}}{\Gamma,\Delta,A\supset B\vdash C}\mathsf{C}$$

$$\equiv_{\mathsf{CP}} := (\equiv \cup \equiv_{\mathsf{c}} \cup \equiv_{\mathsf{e}}) \qquad \equiv_{\mathsf{WIS}} = \equiv_{\lambda} := (\equiv \cup \equiv_{\mathsf{c}} \cup \equiv_{\mathsf{e}} \cup \equiv_{\mathsf{u}})$$

On Constructive Modal Logic

Crash course on Constructive Modal Logic CK

$$A, B ::= 1 \mid a \mid A \supset B \mid A \wedge B$$

Intuitionistic propositional logic (LI)

$$\dfrac{}{a \vdash a}\,\mathsf{AX} \quad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,\supset^{\mathsf{R}} \quad \dfrac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\,\supset^{\mathsf{L}} \quad \dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge^{\mathsf{R}} \quad \dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge^{\mathsf{L}}$$

$$\dfrac{}{\vdash 1}\,1 \quad \dfrac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathsf{C} \quad \dfrac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathsf{W}$$

Crash course on Constructive Modal Logic CK

$$A, B ::= 1 \mid a \mid A \supset B \mid A \wedge B \mid \Box A \mid \Diamond A$$

Intuitionistic propositional logic (LI)

$+$

Nec rule: if F is provable, then $\Box F$ is provable

$+$

$\mathsf{k}_1 : \Box(A \supset B) \supset (\Box A \supset \Box B) \qquad \mathsf{k}_2 : \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$

$$\frac{}{a \vdash a}\,\mathsf{AX} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,\supset^{\mathsf{R}} \quad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\,\supset^{\mathsf{L}} \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge^{\mathsf{R}} \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge^{\mathsf{L}}$$

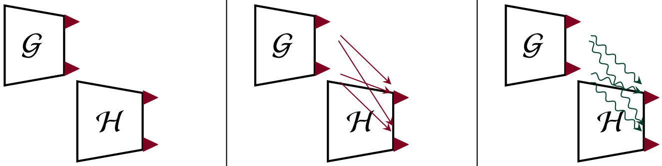
$$\frac{}{\vdash 1}\,1 \quad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathsf{C} \quad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathsf{W} \quad \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\mathsf{K}^{\Box} \quad \frac{A, \Gamma \vdash B}{\Diamond A, \Box\Gamma \vdash \Diamond B}\,\mathsf{K}^{\Diamond}$$

Constructive modal Logic
(Combinatorial Proofs and Game Semantics)

$$\llbracket a \rrbracket = a \qquad \llbracket A \wedge B \rrbracket = \llbracket A \rrbracket + \llbracket B \rrbracket \qquad \llbracket A \supset B \rrbracket = \llbracket A \rrbracket \mathrel{-\!\!\triangleright} \llbracket B \rrbracket$$

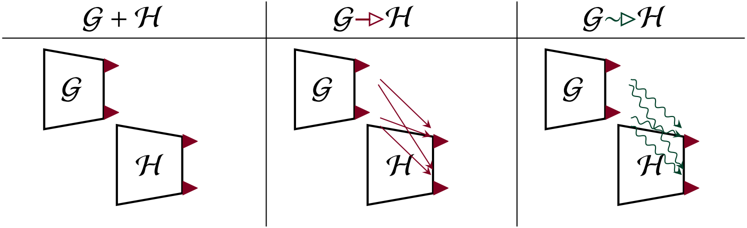
$$\llbracket \Box A \rrbracket = \Box \mathrel{\sim\!\triangleright} \llbracket A \rrbracket \qquad \llbracket \Diamond A \rrbracket = \Diamond \mathrel{\sim\!\triangleright} \llbracket A \rrbracket$$

$\mathcal{G} + \mathcal{H}$	$\mathcal{G} \mathrel{-\!\!\triangleright} \mathcal{H}$	$\mathcal{G} \mathrel{\sim\!\triangleright} \mathcal{H}$



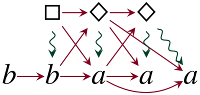
$$[\![a]\!] = a \qquad [\![A \wedge B]\!] = [\![A]\!] + [\![B]\!] \qquad [\![A \supset B]\!] = [\![A]\!] \mathbin{-\!\!\triangleright} [\![B]\!]$$

$$[\![\Box A]\!] = \Box \mathbin{\sim\!\!\triangleright} [\![A]\!] \qquad [\![\Diamond A]\!] = \Diamond \mathbin{\sim\!\!\triangleright} [\![A]\!]$$



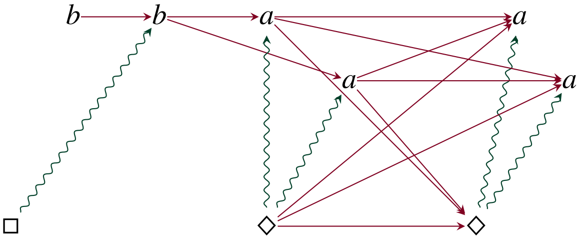
Examples:

$$[\![(\Box(b \supset b) \supset \Diamond a) \supset \Diamond(a \wedge a)]\!] =$$



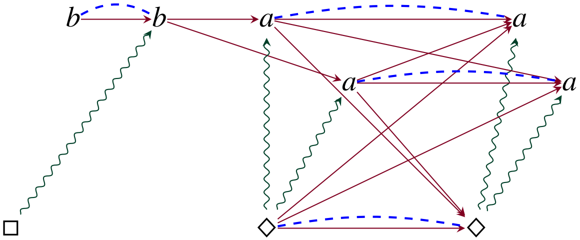
Combinatorial Proofs for CK [AiML2023]:

- Arenas for modal formulas



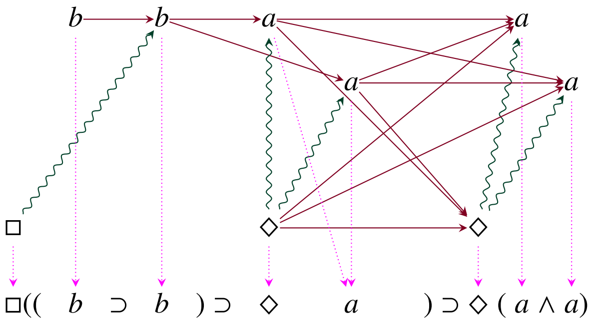
Combinatorial Proofs for CK [AiML2023]:

- Arenas for modal formulas
- Linear proofs = arenas + specific vertices partitions



Combinatorial Proofs for CK [AiML2023]:

- Arenas for modal formulas
- Linear proofs = arenas + specific vertices partitions
- Specific morphisms = deep-WC derivations



Combinatorial Proofs for CK [AiML2023]:

- Arenas for modal formulas
- Linear proofs = arenas + specific vertices partitions
- Specific morphisms = deep-WC derivations
- We can factorize CK and CD proofs

$$\Vert \mathsf{IMLL\text{-}X}^{\bullet}$$

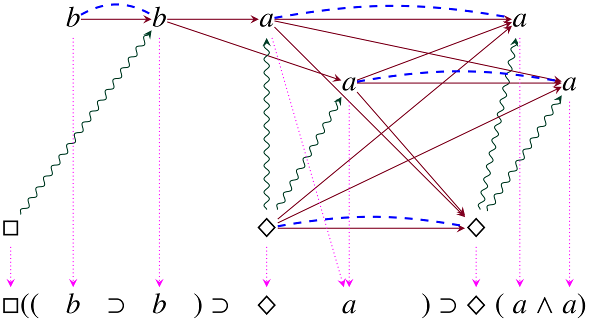
$$\Box((b \supset b) \supset \Diamond (a \wedge a)) \supset \Diamond (a \wedge a)$$

$$\Vert \mathsf{LI}^{\bullet}_{\downarrow}$$

$$\Box((b \supset b) \supset \Diamond a) \supset \Diamond (a \wedge a)$$

Combinatorial Proofs for CK [AiML2023]:

- Arenas for modal formulas
- Linear proofs = arenas + specific vertices partitions
- Specific morphisms = deep-WC derivations
- We can factorize CK and CD proofs
- We have combinatorial proofs for CK and CD!

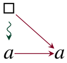


Back to games...

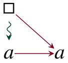
How to play:

- ∘ starts on a root
- any non initial move is *justified* by a previous move
- ∘ is *shortsighted*: his moves points the previous •-move
- each •-move must ”reply” the previous ∘-move

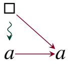
Here I should have no chances to win

Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$\supset^{\mathsf{R}} \dfrac{\dfrac{\mathit{FAIL}}{\Box a \vdash a}}{\vdash \Box a \supset a}$	$\Box a \supset a$ $\mathcal{S} = \{a^{\circ} \quad a^{\bullet}\}$

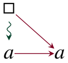
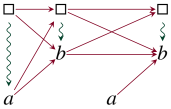
Here I should have no chances to win

Formula	Arena	Derivation (attempt)	WIS
$\square a \supset a$		$\supset^R \dfrac{\dfrac{\mathit{FAIL}}{\square a \vdash a}}{\vdash \square a \supset a}$	$\square a \supset a$ $\mathcal{S} = \{a^\circ \quad \overset{\square^\bullet}{a^\bullet}\}$

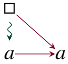
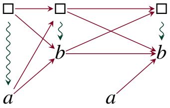
Here I should have no chances to win

Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$\supset^{R}\ \dfrac{\dfrac{\mathit{FAIL}}{\Box a \vdash a}}{\vdash \Box a \supset a}$	$\Box a \supset a$ $\mathcal{S} = \{\overset{\epsilon}{a^{\circ}} \quad \overset{\Box^{\bullet}}{a^{\bullet}}\}$

Here I should have no chances to win

Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$\supset^{\mathsf{R}}\ \dfrac{\dfrac{\mathit{FAIL}}{\Box a \vdash a}}{\vdash \Box a \supset a}$	$\Box a \supset a$ $\mathcal{S} = \{\overset{\epsilon}{a^\circ}\quad \overset{\Box^\bullet}{a^\bullet}\}$
$(\Box a \supset \Box b) \supset \Box(a \supset b)$		$\supset^{\mathsf{R}}\ \dfrac{\dfrac{\dfrac{\mathit{FAIL}}{\vdash \Box a} \qquad \dfrac{\Vert}{\Box b \vdash \Box(a \supset b)}}{\Box a \supset \Box b \vdash \Box(a \supset b)}}{\vdash (\Box a \supset \Box b) \supset \Box(a \supset b)}$	$(\Box a \supset \Box b) \supset \Box(a \supset b)$ $\mathcal{S} = \{\ \overset{\Box^\circ}{b^\circ}\quad \overset{\Box^\bullet}{b^\bullet}\quad \overset{\Box^\circ}{a^\circ}\quad \overset{\Box^\circ}{a^\bullet}\}$

Here I should have no chances to win

Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$\dfrac{\dfrac{FAIL}{\Box a \vdash a}}{\vdash \Box a \supset a}\supset^{R}$	$\Box a \supset a$ $\mathcal{S} = \{\overset{\epsilon}{a^{\circ}} \quad \overset{\Box^{\bullet}}{a^{\bullet}}\}$
$(\Box a \supset \Box b) \supset \Box(a \supset b)$		$\dfrac{\dfrac{\dfrac{FAIL}{\vdash \Box a} \qquad \dfrac{\Vert}{\Box b \vdash \Box(a \supset b)}}{\Box a \supset \Box b \vdash \Box(a \supset b)}}{\vdash (\Box a \supset \Box b) \supset \Box(a \supset b)}\supset^{R}$	$(\Box a \supset \Box b) \supset \Box(a \supset b)$ $\mathcal{S} = \{ \quad \overset{\Box^{\circ}}{b^{\circ}} \quad \overset{\Box^{\bullet}}{b^{\bullet}} \quad \overset{\Box^{\circ}}{a^{\circ}} \quad \overset{\Box^{\circ}}{a^{\bullet}}\}$

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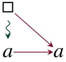
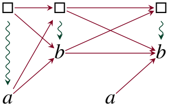
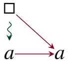
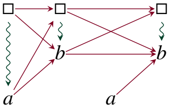
Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$\supset^R \dfrac{\dfrac{FAIL}{\Box a \vdash a}}{\vdash \Box a \supset a}$	$\Box a \supset a$ $\mathcal{S} = \{ \overset{\epsilon}{a^\circ} \quad \overset{\Box^\bullet}{a^\bullet} \}$
$(\Box a \supset \Box b) \supset \Box(a \supset b)$		$\supset^R \dfrac{\dfrac{\dfrac{FAIL}{\vdash \Box a} \qquad \dfrac{\Vert}{\Box b \vdash \Box(a \supset b)}}{\Box a \supset \Box b \vdash \Box(a \supset b)}}{\vdash (\Box a \supset \Box b) \supset \Box(a \supset b)}$	$(\Box a \supset \Box b) \supset \Box(a \supset b)$ $\mathcal{S} = \{ \overset{\Box^\circ}{b^\circ} \quad \overset{\Box^\bullet}{b^\bullet} \quad \overset{\Box^\circ - - - \Box^\circ}{a^\circ \qquad a^\bullet} \}$

Here I should have no chances to win

Formula	Arena	Derivation (attempt)	WIS
$\Box a \supset a$		$\supset^{R}\dfrac{\dfrac{FAIL}{\Box a \vdash a}}{\vdash \Box a \supset a}$	$\Box a \supset a$ $\mathcal{S} = \{\overset{\epsilon}{a^\circ} \quad \overset{\Box^\bullet}{a^\bullet}\}$
$(\Box a \supset \Box b) \supset \Box(a \supset b)$		$\supset^{R}\dfrac{\dfrac{\dfrac{FAIL}{\vdash \Box a} \qquad \dfrac{\Vert}{\Box b \vdash \Box(a \supset b)}}{\Box a \supset \Box b \vdash \Box(a \supset b)}}{\vdash (\Box a \supset \Box b) \supset \Box(a \supset b)}$	$(\Box a \supset \Box b) \supset \Box(a \supset b)$ $\mathcal{S} = \{ \overset{\Box^\circ}{b^\circ} \quad \overset{\Box^\bullet}{b^\bullet} \quad \overset{\Box^\circ}{a^\circ} \quad \overset{\Box^\circ}{a^\bullet}\}$

Theorem (Full Completeness)

Every CK-WIS *on* $[\![F]\!]$ *is the image of a proof of* F*.*

Additional conditions on views [Tableaux2021]:

1. no $\square$ occurs;
2. each $\bullet$-move is at the same "height" of the previous $\circ$-move;
3. each $\sim$-class contains a unique $\circ$-vertex;
4. each $\sim$-class contains a (unique) $\Diamond^{\circ}$ iff it contains a unique $\Diamond^{\bullet}$.

Relation between CK-WISs and modal λ-terms

In intuitionistic logic we have a 1-to-1 correspondence

$$\left\{\eta\beta\text{-normal } \lambda\text{-terms}\right\} \leftrightarrow \left\{\text{WISs}\right\}$$

which cannot be extended!

Problem: even in a "minimal" λ-calculus

$$t := x \mid \lambda x.t \mid (t)u \mid \text{Let } \vec{x} \text{ be } \vec{u} \text{ in } t$$

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$$t := x \mid \lambda x.t \mid (t)u \mid t\left[\vec{t}/\vec{x}\right]$$

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$$x[t/y, t/y] \simeq x[t/y]$$

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$$x[t/y, t/y] \simeq x[t/y]$$

Solution: additional reductions

$$M\left[\vec{P}, N, \vec{Q}/\vec{x}, y, \vec{z}\right]_{\blacksquare} \rightsquigarrow_{\kappa} M\left[\vec{P}, \vec{Q}/\vec{x}, \vec{z}\right]_{\blacksquare} \quad \text{if no } y \text{ in } M$$

$$M\left[\vec{P}, N, N, \vec{Q}/\vec{x}, y_1, y_2, \vec{z}\right]_{\blacksquare} \rightsquigarrow_{\kappa} M\{v, v/y_1, y_2\}\left[\vec{P}, N, \vec{Q}/\vec{x}, v, \vec{z}\right]_{\blacksquare} \quad v \text{ fresh}$$

Relation between CK-WISs and modal λ-terms

For constructive modal logic we have a 1-to-1 correspondence [ArXiv23]

$$\left\{\eta\beta\kappa\text{-normal } \lambda\text{-terms}\right\} \leftrightarrow \left\{\text{WISs}\right\}$$

Problem: even in a "minimal" λ-calculus

$$t \coloneqq x \mid \lambda x.t \mid (t)u \mid t\left[\vec{t}/\vec{x}\right]$$

$$x[t/y, t/y] \simeq x[t/y]$$

Solution: additional reductions

$$M\left[\vec{P}, N, \vec{Q}/\vec{x}, y, \vec{z}\right]_{\blacksquare} \rightsquigarrow_{\kappa} M\left[\vec{P}, \vec{Q}/\vec{x}, \vec{z}\right]_{\blacksquare} \quad \text{if no } y \text{ in } M$$

$$M\left[\vec{P}, N, N, \vec{Q}/\vec{x}, y_1, y_2, \vec{z}\right]_{\blacksquare} \rightsquigarrow_{\kappa} M\{v, v/y_1, y_2\}\left[\vec{P}, N, \vec{Q}/\vec{x}, v, \vec{z}\right]_{\blacksquare} \quad v \text{ fresh}$$

Proof equivalence in Constructive Modal Logic

Independent rules	$\dfrac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$ $\quad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1$ $\quad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$
Resource Management	$\dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A,B\vdash C}2\times\mathsf{C}}{\Gamma,A\wedge B\vdash C}\wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}2\times\wedge^{\mathsf{L}}}{\Gamma,A\vdash B}\mathsf{C}$ $\quad \dfrac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}2\times\mathsf{W}}{\Gamma,A\wedge B\vdash C}\wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}\mathsf{C}}{\Gamma,A,A\vdash B}\mathsf{W} \equiv_{\mathsf{c}} \Gamma,A,A\vdash B$ $\quad \dfrac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}\mathsf{W}}{\Gamma,A\vdash B}\mathsf{C} \equiv_{\mathsf{c}} \Gamma,A\vdash B$
Excising and Unfolding	$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta\vdash C}{B,\Delta\vdash C}\mathsf{W}}{\Gamma,\Delta,A\supset B\vdash C}\supset^{\mathsf{L}} \equiv_{\mathsf{e}} \dfrac{\Delta\vdash C}{\Gamma,\Delta,A\supset B\vdash C}\mathsf{W}$ $\quad \dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}\mathsf{C}}{\Gamma,A\supset B\vdash C}\supset^{\mathsf{L}} \equiv_{\mathsf{u}} \dfrac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\supset^{\mathsf{L}}}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\supset^{\mathsf{L}}}{\Gamma,\Delta,A\supset B\vdash C}\mathsf{C}$
Structural vs K	$\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}\mathsf{W}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{K}^{\Box} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma\vdash A}{\Box\Gamma\vdash\Box A}\mathsf{K}^{\Box}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{W}$ $\quad \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Gamma,B\vdash A}\mathsf{C}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{K}^{\Box} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Box\Gamma,\Box B,\Box B\vdash\Box A}\mathsf{K}^{\Box}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{C}$ $\dfrac{\dfrac{\Gamma,B\vdash A}{\Gamma,B,C\vdash A}\mathsf{W}}{\Box\Gamma,\Diamond B,\Box C,\vdash\Box A}\mathsf{K}^{\Diamond} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B\vdash A}{\Box\Gamma,\Diamond B\vdash\Diamond A}\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Box C\vdash\Box A}\mathsf{W}$ $\quad \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Gamma,B,C\vdash A}\mathsf{C}}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}\mathsf{K}^{\Box} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Box\Gamma,\Diamond B,\Box C,\Box C\Box\vdash\Diamond A}\mathsf{K}^{\Box}}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}\mathsf{C}$
Jumps	$\dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}\mathsf{W}}{\Box\Gamma,\Diamond B\vdash\Diamond A}\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}\mathsf{W} \equiv_{\Diamond\mathsf{w}} \dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,C\vdash A}\mathsf{W}}{\Box\Gamma,\Diamond C\vdash\Diamond A}\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}\mathsf{W}$

$$\equiv_{\mathsf{CP}} := (\equiv \cup \equiv_{\mathsf{c}} \cup \equiv_{\mathsf{e}}) \qquad \equiv_{\lambda} := (\equiv_{\mathsf{CP}} \cup \equiv_{\mathsf{u}}) \qquad \equiv_{\mathsf{WIS}} := (\equiv_{\lambda} \cup \equiv_{\Box\mathsf{c}}) \qquad \equiv_{\Diamond\mathsf{w}} := (\equiv_{\mathsf{WIS}} \cup \equiv_{\Box\mathsf{c}})$$

Sum up (Constructive Modal Logic):

- Sequent calculus
  - proof systems [Cook-Reckhow]
  - no proof equivalence
  - Compositionality via cut
- Combinatorial proofs
  - proof systems [Cook-Reckhow]
  - (resource-sensitive) proof equivalence
  - Compositionality under study
- Old λ-calculus / Natural Deduction
  - some expected equivalences seems to be missed
  - No 1-to-1 correspondence between CK-WISs and $\eta\beta$-normal λ-terms
- Winning Innocent Strategies / New λ-calculus
  - Full-complete concrete model for denotational semantics
  - Not a proof system
  - (not resource sensitive) proof equivalence
  - 1-to-1 correspondence between CK-WISs and $\eta\beta\kappa$-normal λ-terms
- Structural Rules and Modalities interact weirdly (P-space complexity)

No possible proof systems capturing the whole proof equivalence

Related works/Works in Progress:

- Combinatorial Proofs and Game Semantics for CS4
- Combinatorial Proofs as proof certificates (with modules)
- Combinatorial Proofs Normalization
- Extend results on λ-calculus for CK (include $\diamond$ and $\wedge$)
- Re-study categorical semantics (!)

Thanks

Thanks

Questions?