Infinitary cut-elimination via finite approximations (extended version)

Matteo Acclavio
University of Southern Denmark, Denmark

Gianluca Curzi
University of Birmingham, UK

Giulio Guerrieri
Aix Marseille Univ, CNRS, LIS UMR 7020, Marseille, France

Abstract

We investigate non-wellfounded proof systems based on parsimonious logic, a weaker variant of linear logic where the exponential modality ! is interpreted as a constructor for streams over finite data. Logical consistency is maintained at a global level by adapting a standard progressing criterion. We present an infinitary version of cut-elimination based on finite approximations, and we prove that, in presence of the progressing criterion, it returns well-defined non-wellfounded proofs at its limit. Furthermore, we show that cut-elimination preserves the progressive criterion and various regularity conditions internalizing degrees of proof-theoretical uniformity. Finally, we provide a denotational semantics for our systems based on the relational model.

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1 Introduction

Non-wellfounded proof theory studies proofs as possibly infinite (but finitely branching) trees, where logical consistency is maintained via global conditions called *progressing* (or *validity*) *criteria*. In this setting, the so-called *regular* (also called *circular*) proofs receive a special attention, as they admit a finite description in terms of (possibly cyclic) directed graphs.

This area of proof theory makes its first appearance (in its modern guise) in the modal μ-calculus [28, 13]. Since then, it has been extensively investigated from many perspectives (see, e.g., [7, 33, 12, 21]), establishing itself as an ideal setting for manipulating least and greatest fixed points, and hence for modeling induction and coinduction principles.

Non-wellfounded proof theory has been applied to constructive fixed point logics i.e., with a computational interpretation based on the *Curry-Howard correspondence* [34]. A key example can be found in the context of *linear logic* (LL) [19], a logic implementing a finer control on resources thanks to the *exponential* modalities ! and ?. In this framework, the most extensively studied fixed point logic is μMALL, defined as the exponential-free fragment of LL with least and greatest fixed point operators (respectively, μ and its dual ν) [6, 5].

In [6] Baelde and Miller have shown that the exponentials can be recovered in μMALL by exploiting the fixed points operators, i.e., by defining $!A := \nu X.(\mathbf{1} \,\&\, A \,\&\, (X \otimes X))$ and $?A := \mu X.(\bot \oplus A \oplus (X \parr X))$. As these authors notice, the fixed point-based definition of !

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and ? can be regarded as a more permissive variant of the standard exponentials, since a proof of $\nu X.(\mathbf{1} \& A \& (X \otimes X))$ could be constructed using different proofs of A, whereas in LL a proof of $!A$ is constructed uniformly using a single proof of A. This proof-theoretical notion of *non-uniformity* is indeed a central feature of the fixed-point exponentials.

However, the above encoding is not free of issues. First, as discussed in full detail in [16], the encoding of the exponentials does not verify the Seely isomorphisms, syntactically expressed by the equivalence $!(A \& B) \multimap\!\!\circ (!A \otimes !B)$, an essential property for modeling exponentials in LL. Specifically, the fixed-point definition of ! relies on the multiplicative connective $\otimes$, which forces an interpretation of $!A$ based on lists rather than multisets. Secondly, as pointed out in [6], there is a neat mismatch between cut-elimination for the exponentials of LL and the one for the fixed point exponentials of μMALL. While the first problem is related to syntactic deficiencies of the encoding, and does not undermine further investigations on fixed point-based definitions of the exponential modalities, the second one is more critical. These apparent differences between the two exponentials contribute to stressing an important aspect in linear logic modalities, i.e., their *non-canonicity* [30, 11][1].

On a parallel research thread, Mazza [24, 25, 26] studied *parsimonious logic*, a variant of linear logic where the exponential modality ! satisfies Milner's law (i.e., $!A \multimap\!\!\circ A \otimes !A$) and invalidates the implications $!A \multimap !!A$ (*digging*) and $!A \multimap !A \otimes !A$ (*contraction*). In parsimonious logic, a proof of $!A$ can be interpreted as a *stream* over (a finite set of) proofs of A, i.e., as a greatest fixed point, where the linear implications $A \otimes !A \multimap !A$ (*co-absorption*) and $!A \multimap A \otimes !A$ (*absorption*) can be read computationally as the *push* and *pop* operations on streams. More specifically, a formula $!A$ is introduced by an *infinitely branching rule* that takes a finite set of proofs $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of A and a (possibly non-recursive) function $f : \mathbb{N} \to \{1, \ldots, n\}$ as premises, and constructs a proof of $!A$ representing a stream of proofs of the form $\mathfrak{S} = (\mathcal{D}_{f(0)}, \mathcal{D}_{f(1)}, \ldots, \mathcal{D}_{f(n)}, \ldots)$. Hence, parsimonious logic exponential modalities exploit in an essential way the above-mentioned proof-theoretical non-uniformity, which in turn deeply interfaces with notions of non-uniformity from computational complexity [26].

The analysis of parsimonious logic conducted in [25, 26] reveals that fixed point definitions of the exponentials are better behaving when digging and contraction are discarded. On the other hand, the co-absorption rule cannot be derived in LL, and so it prevents parsimonious logic becoming a genuine subsystem of the latter. This led the authors of the present paper to introduce *parsimonious linear logic*, a subsystem of linear logic (in particular, co-absorption-free) that nonetheless allows a stream-based interpretation of the exponentials.

We present two finitary proof systems for parsimonious linear logic: the system nuPLL, supporting non-uniform exponentials, and PLL, a fully uniform version. We investigate non-wellfounded counterparts of nuPLL and PLL, adapting to our setting the progressing criterion to maintain logical consistency. To recover the proof-theoretical behavior of nuPLL and PLL, we identify further global conditions on non-wellfounded proofs, that is, some forms of regularity to capture the notions of uniformity and non-uniformity. This leads us to two main non-wellfounded proof systems: *regular parsimonious linear logic* (rPLL^∞), defined via the regularity condition and corresponding to PLL, and *weakly regular parsimonious linear logic* (wrPLL^∞), defined via a *weak regularity* condition and corresponding to nuPLL.

The major contribution of this paper is the study of continuous cut-elimination in the setting of non-wellfounded parsimonious linear logic. We first introduce Scott-domains of partially defined non-wellfounded proofs, ordered by an approximation relation. Then, we define special infinitary proof rewriting strategies called *maximal and continuous infinitary*

1 One can construct LL proof systems with alternative (not equivalent) exponential modalities, see [27].

cut-elimination strategies (mc-ices) which compute (Scott-)continuous functions. Productivity in this framework is established by showing that, in presence of a good global condition (progressing, regularity or weak regularity), these continuous functions return totally defined cut-free non-wellfounded proofs and preserve the global condition: progressing (Theorem 40), and regularity or weak regularity (Theorem 48).

On a technical side, we stress that our methods and results distinguish from previous approaches to cut-elimination in a non-wellfounded setting in many respects. First, we get rid of many technical notions typically introduced to prove infinitary cut-elimination, such as the *multicut rule* or the *fairness conditions* (as in, e.g., [18, 5]), as these notions are subsumed by a *finitary approximation* approach to cut-elimination. Furthermore, we prove productivity of cut-elimination and preservation of progressiveness in a more direct and constructive way, i.e., without going through auxiliary proof systems and avoiding arguments by contradiction (see, e.g., [5]). Finally, we prove for the first time preservation of regularity properties under continuous cut-elimination, essentially exploiting methods for compressing transfinite rewriting sequences to ω-long ones from [35, 24, 32].

Finally, we define a denotational semantics for non-wellfounded parsimonious logic based on the relational model, with a standard multiset-based interpretation of the exponentials, and we show that this semantics is preserved under continuous cut-elimination (Theorem 56). We also prove that extending non-wellfounded parsimonious linear logic with digging prevents the existence of a cut-elimination result preserving the semantics (Theorem 58). Therefore, the impossibility of a stream-based definition of ! that validates digging (and contraction).

2 Preliminary notions

In this section we recall some basic notions from (non-wellfounded) proof theory, fixing the notation that will be adopted in this paper.

2.1 Derivations and coderivations

We assume that the reader is familiar with the syntax of sequent calculus, e.g. [36]. Here we specify some conventions adopted to simplify the content of this paper.

We consider (**sequent**) **rules** of the form $\mathsf{r}\frac{}{\Gamma}$ or $\mathsf{r}\frac{\Gamma_1}{\Gamma}$ or $\mathsf{r}\frac{\Gamma_1 \quad \Gamma_2}{\Gamma}$, and we refer to the sequents Γ_1 and Γ_2 as the **premises**, and to the sequent Γ as the **conclusion** of the rule r. To avoid technicalities of the sequents-as-lists presentation, we follow [5] and we consider **sequents** as *sets of occurrences of formulas* from a given set of formulas. In particular, when we refer to a formula in a sequent we always consider a *specific occurrence* of it.

▶ **Definition 1.** *A (binary, possibly infinite)* ***tree*** $\mathcal{T}$ *is a subset of words in* $\{1,2\}^*$ *that contains the empty word* ϵ *(the* ***root*** *of* $\mathcal{T}$*) and is* ordered-prefix-closed *(i.e., if* $n \in \{1,2\}$ *and* $vn \in \mathcal{T}$*, then* $v \in \mathcal{T}$*, and if moreover* $v2 \in \mathcal{T}$*, then* $v1 \in \mathcal{T}$*). The elements of* $\mathcal{T}$ *are called* ***nodes*** *and their* ***height*** *is the length of the word. A* ***child*** *of* $v \in \mathcal{T}$ *is any* $vn \in \mathcal{T}$ *with* $n \in \{1,2\}$*. The* ***prefix order*** *is a partial order* $\leq_{\mathcal{T}}$ *on* $\mathcal{T}$ *defined by: for any* $v, v' \in \mathcal{T}$*,* $v \leq_{\mathcal{T}} v'$ *if* $v' = vw$ *for some* $w \in \{1,2\}^*$*. A maximal element of* $\leq_{\mathcal{T}}$ *is a* ***leaf*** *of* $\mathcal{T}$*. A* ***branch*** *of* $\mathcal{T}$ *is a set* $\mathcal{B} \subseteq \mathcal{T}$ *such that* $\epsilon \in \mathcal{B}$ *and if* $w \in \mathcal{B}$ *is not a leaf of* $\mathcal{T}$ *then* w *has exactly one child in* $\mathcal{B}$*.*

A ***coderivation*** *over a set of rules* $\mathcal{S}$ *is a labeling* $\mathcal{D}$ *of a tree* $\mathcal{T}$ *by sequents such that if* v *is a node of* $\mathcal{T}$ *with children* $v_1, \ldots, v_n$ *(with* $n \in \{0,1,2\}$*), then there is an occurrence of a rule* r *in* $\mathcal{S}$ *with conclusion the sequent* $\mathcal{D}(v)$ *and premises the sequents* $\mathcal{D}(v_1), \ldots, \mathcal{D}(v_n)$*. The* ***height*** *of* r *in* $\mathcal{D}$ *is the height of the node* $v \in \mathcal{T}$ *such that* $\mathcal{D}(v)$ *is the conclusion of* r*.*

$$\mathsf{ax}\,\frac{}{A, A^{\perp}} \quad \mathsf{cut}\,\frac{\Gamma, A \quad A^{\perp}, \Delta}{\Gamma, \Delta} \quad \otimes\,\frac{\Gamma, A \quad B, \Delta}{\Gamma, \Delta, A \otimes B} \quad ⅋\,\frac{\Gamma, A, B}{\Gamma, A ⅋ B} \quad 1\,\frac{}{1} \quad \perp\,\frac{\Gamma}{\Gamma, \perp} \quad \mathsf{f!p}\,\frac{\Gamma, A}{?\Gamma, !A} \quad ?\mathsf{w}\,\frac{\Gamma}{\Gamma, ?A} \quad ?\mathsf{b}\,\frac{\Gamma, A, ?A}{\Gamma, ?A}$$

Figure 1 Sequent calculus rules of PLL.

*The **conclusion** of $\mathcal{D}$ is the sequent $\mathcal{D}(\epsilon)$. If v is a node of the tree, the **sub-coderivation** of $\mathcal{D}$ rooted at v is the coderivation $\mathcal{D}_v$ defined by $\mathcal{D}_v(w) = \mathcal{D}(vw)$.*

A coderivation $\mathcal{D}$ is r-***free*** *(for a rule* r $\in \mathcal{S}$*) if it contains no occurrence of* r*. It is **regular** if it has finitely many distinct sub-coderivations; it is **non-wellfounded** if it labels an infinite tree, and it is a **derivation** (with **size** $|\mathcal{D}| \in \mathbb{N}$) if it labels a finite tree (with $|\mathcal{D}|$ nodes).*

Given a set of coderivations X*, a sequent Γ is **provable** in* X *(noted $\vdash_{\mathsf{X}} \Gamma$) if there is a coderivation in* X *with conclusion Γ.*

While derivations are usually represented as finite trees, regular coderivations can be represented as *finite* directed (possibly cyclic) graphs: a cycle is created by linking the roots of two identical subcoderivations.

▶ **Definition 2.** *Let $\mathcal{D}$ be a coderivation labeling a tree $\mathcal{T}$. A **bar** (resp. **prebar**) of $\mathcal{D}$ is a set $\mathcal{V} \subseteq \mathcal{T}$ where:*

- *any branch (resp. infinite branch) of the tree $\mathcal{T}$ underlying $\mathcal{D}$ contains a node in $\mathcal{V}$;*
- *any pair of nodes in $\mathcal{V}$ are mutually incomparable with respect to the prefix order $\leq_{\mathcal{T}}$.*

*The **height** of a prebar $\mathcal{V}$ of $\mathcal{D}$ is the minimal height of the nodes of $\mathcal{V}$.*

3 Parsimonious Linear Logic

In this paper we consider the set of **formulas** for propositional multiplicative-exponential linear logic with units (MELL). These are generated by a countable set of propositional variables $\mathcal{A} = \{X, Y, \dots\}$ using the following grammar:

$$A, B ::= X \mid X^{\perp} \mid A \otimes B \mid A ⅋ B \mid !A \mid ?A \mid 1 \mid \perp$$

A **!-formula** (resp. **?-formula**) is a formula of the form $!A$ (resp. $?A$). **Linear negation** $(\cdot)^{\perp}$ is defined by De Morgan's laws $(A^{\perp})^{\perp} = A$, $(A \otimes B)^{\perp} = A^{\perp} ⅋ B^{\perp}$, $(!A)^{\perp} = ?A^{\perp}$, and $(1)^{\perp} = \perp$ while **linear implication** is defined as $A \multimap B := A^{\perp} ⅋ B$.

▶ **Definition 3.** ***Parsimonious linear logic**, denoted by* PLL*, is the set of rules in Figure 1, that is, **axiom** (*ax*), **cut** (*cut*), **tensor** ($\otimes$), **par** (⅋), **one** (1), **bottom** ($\perp$), **functorial promotion** (*f!p*), **weakening** (*?w*), **absorption** (*?b*). Rules* ax*, $\otimes$, ⅋, 1 and $\perp$ are called **multiplicative**, while rules* f!p*,* ?w *and* ?b *are called **exponential**. We also denote by* PLL *the set of derivations over the rules in* PLL*.*

▶ **Example 4.** Figure 2 gives some examples of derivation in PLL. The (distinct) derivations $\underline{0}$ and $\underline{1}$ prove the same formula $\mathbf{N} := !(X \multimap X) \multimap X \multimap X$. The derivation $\mathcal{D}_{\mathsf{abs}}$ proves the *absorption law* $!A \multimap A \otimes !A$; the derivation $\mathcal{D}_{\mathsf{der}}$ proves the *dereliction law* $!A \multimap A$.

The **cut-elimination** relation $\to_{\mathsf{cut}}$ in PLL is the union of **principal** cut-elimination steps in Figure 3 (**multiplicative**) and Figure 4 (**exponential**) and **commutative** cut-elimination steps in Figure 5. The reflexive-transitive closure of $\to_{\mathsf{cut}}$ is noted $\to_{\mathsf{cut}}^{*}$.

▶ **Theorem 5.** *For every $\mathcal{D} \in$ PLL, there is a cut-free $\mathcal{D}' \in$ PLL such that $\mathcal{D} \to_{\mathsf{cut}}^{*} \mathcal{D}'$.*

$\underline{0}$	$\underline{1}$	$\mathcal{D}_{\mathsf{abs}}$	$\mathcal{D}_{\mathsf{der}}$
$\dfrac{\dfrac{\dfrac{\dfrac{}{X^\perp, X}\,\mathsf{ax}}{?(X \otimes X^\perp), X^\perp, X}\,?\mathsf{w}}{?(X \otimes X^\perp), X^\perp \parr X}\,\parr}{?(X \otimes X^\perp) \parr X^\perp \parr X}\,\parr$	$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X^\perp, X}\,\mathsf{ax} \quad \dfrac{}{X^\perp, X}\,\mathsf{ax}}{X \otimes X^\perp, X^\perp, X}\,\otimes}{?(X \otimes X^\perp), X \otimes X^\perp, X^\perp, X}\,?\mathsf{w}}{?(X \otimes X^\perp), X^\perp, X}\,?\mathsf{b}}{?(X \otimes X^\perp) \parr X^\perp \parr X}\,\parr\times 2$	$\dfrac{\dfrac{\dfrac{\dfrac{}{A^\perp, A}\,\mathsf{ax} \quad \dfrac{}{?A^\perp, !A}\,\mathsf{ax}}{A^\perp, ?A^\perp, A \otimes !A}\,\otimes}{?A^\perp, A \otimes !A}\,?\mathsf{b}}{?A^\perp \parr (A \otimes !A)}\,\parr$	$\dfrac{\dfrac{\dfrac{\dfrac{}{A^\perp, A}\,\mathsf{ax}}{A^\perp, ?A^\perp, A}\,?\mathsf{w}}{?A^\perp, A}\,?\mathsf{b}}{?A^\perp \parr A}\,\parr$

Figure 2 Examples of derivations in PLL.

$$\dfrac{\dfrac{}{A, A^\perp}\,\mathsf{ax} \quad \Gamma, A}{\Gamma, A}\,\mathsf{cut} \;\to_{\mathsf{cut}}\; \Gamma, A \qquad \dfrac{\dfrac{\Gamma, A, B}{\Gamma, A \parr B}\,\parr \quad \dfrac{\Delta, A^\perp \quad B^\perp, \Sigma}{\Delta, A^\perp \otimes B^\perp, \Sigma}\,\otimes}{\Gamma, \Delta, \Sigma}\,\mathsf{cut} \;\to_{\mathsf{cut}}\; \dfrac{\dfrac{\Gamma, B, A \quad A^\perp, \Delta}{\Gamma, \Delta, B}\,\mathsf{cut} \quad B^\perp, \Sigma}{\Gamma, \Delta, \Sigma}\,\mathsf{cut} \qquad \dfrac{\dfrac{\Gamma}{\Gamma, \bot}\,\bot \quad \dfrac{}{1}\,1}{\Gamma}\,\mathsf{cut} \;\to_{\mathsf{cut}}\; \Gamma$$

Figure 3 Multiplicative cut-elimination steps in PLL.

Sketch of proof. We associate with any derivation $\mathcal{D}$ in PLL a derivation $\mathcal{D}^\spadesuit$ in MELL sequent calculus. Thanks to additional commutative cut-elimination steps, we prove that cut-elimination in MELL rewrites $\mathcal{D}^\spadesuit$ to the translation of a derivation in PLL. The termination of cut-elimination in PLL follows from strong normalisation of (second-order) MELL [29]. ◀

Akin to light linear logic [20, 22, 31], the exponential rules of PLL are weaker than those in MELL: the usual promotion rule is replaced by f!p (*functorial promotion*), and the usual contraction and dereliction rules by ?b. As a consequence, the *digging* formula $!A \multimap !!A$ and the *contraction* formula $!A \multimap !A \otimes !A$ are not provable in PLL (unlike the dereliction formula, Example 4). This allows us to interpret computationally these weaker exponentials in terms of streams, as well as to control the complexity of cut-elimination [25, 26].

It is easy to show that MELL = PLL + digging: if we add the digging formula as an axiom (or equivalently, the *digging rule* ??d in Figure 13) to the set of rules in Figure 1, then the contraction formula becomes provable, and the obtained proof system coincides with MELL.

4 Non-wellfounded Parsimonious Linear Logic

In linear logic, a formula $!A$ is interpreted as the availability of A at will. This intuition still holds in PLL. Indeed, the Curry-Howard correspondence interprets rule f!p introducing the modality ! as an operator taking a derivation $\mathcal{D}$ of A and creating a (infinite) *stream* $(\mathcal{D}, \mathcal{D}, \dots, \mathcal{D}, \dots)$ of copies of the proof $\mathcal{D}$. Each element of the stream is accessed via the cut-elimination step f!p vs ?b in Figure 4: rule ?b is interpreted as an operator *popping* one copy of $\mathcal{D}$ out of the stream. Pushing these ideas further, Mazza [25] introduced *parsimonious logic* **PL**, a type system (comprising rules f!p and ?b) characterizing the logspace decidable problems.

Mazza and Terui then introduced in [26] another type system, $\mathbf{nuPL}_{\forall\ell}$, based on parsimonious logic and capturing the complexity class **P**/poly (i.e., the problems decidable by polynomial size families of Boolean circuits [4]). Their system is endowed with a *non-uniform* version of the functorial promotion, which takes a finite set of proofs $\mathcal{D}_1, \dots, \mathcal{D}_n$ of A and a (possibly non-recursive) function $f : \mathbb{N} \to \{1, \dots, n\}$ as premises, and constructs a proof of $!A$ modeling the stream $(\mathcal{D}_{f(0)}, \mathcal{D}_{f(1)}, \dots, \mathcal{D}_{f(n)}, \dots)$. This typing rule is the key tool to encode the so-called *advices* for Turing machines, an essential step to show completeness for **P**/poly.

In a similar vein, we can endow PLL with a non-uniform version of f!p called **infinitely branching promotion** (ib!p), which constructs a stream $(\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_n, \dots)$ with finite

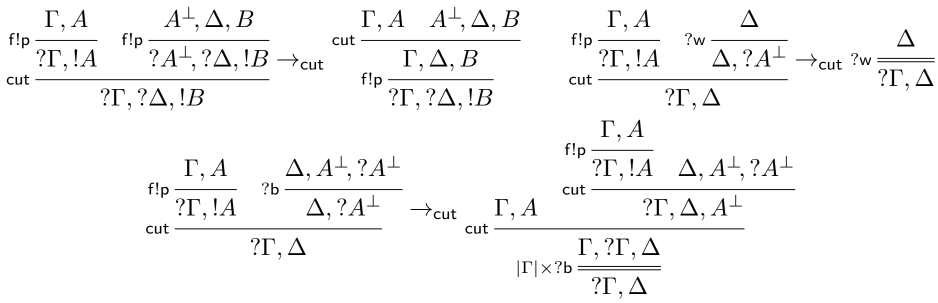


Figure 4 Exponential cut-elimination steps in PLL.

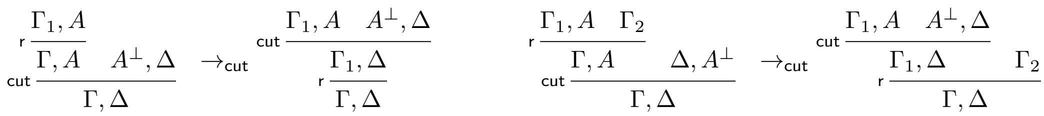


Figure 5 Commutative cut-elimination steps in PLL, where $\mathsf{r} \neq \mathsf{cut}$.

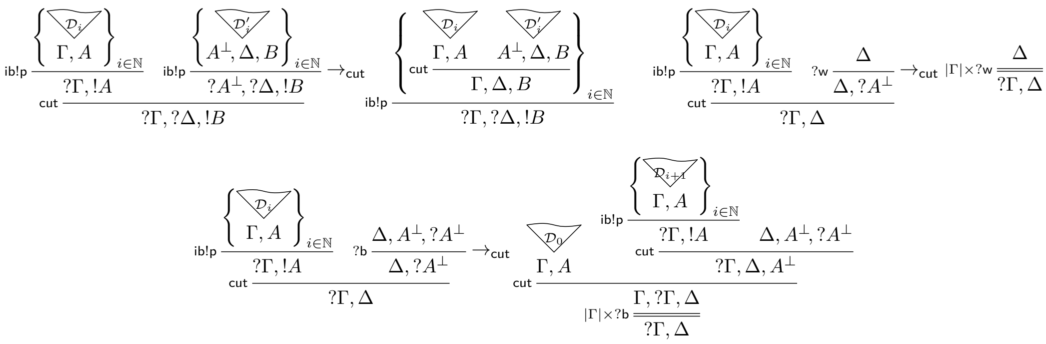
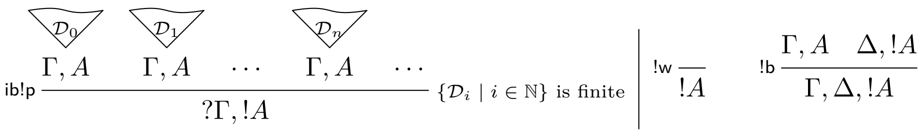


Figure 6 Exponential cut-elimination steps in nuPLL.

support, i.e., made of *finitely* many distinct derivations (of the same conclusion):[2]



(1)

The side condition on ib!p provides a proof theoretic counterpart to the function $f\colon \mathbb{N} \to \{1, \dots, n\}$ in $\mathbf{nuPL}_{\forall\ell}$. Clearly, f!p is subsumed by the rule ib!p, as it corresponds to the special (uniform) case where $\mathcal{D}_i = \mathcal{D}_{i+1}$ for all $i \in \mathbb{N}$.

▶ **Definition 6.** *We define the set of rules* nuPLL $:= \{\mathsf{ax}, \otimes, ⅋, 1, \bot, \mathsf{cut}, ?\mathsf{b}, ?\mathsf{w}, \mathsf{ib!p}\}$. *We also denote by* nuPLL *the set of derivations over the rules in* nuPLL.[3]

There are some notable differences between nuPLL and Mazza and Terui's original system $\mathbf{nuPL}_{\forall\ell}$ [26]. As opposed to nuPLL, $\mathbf{nuPL}_{\forall\ell}$ is formulated as an intuitionistic (type) system.

[2] Rule ib!p is reminiscent of the ω-rule used in (first-order) Peano arithmetic to derive formulas of the form $\forall x\phi$ that cannot be proven in a uniform way.

[3] To be rigorous, this requires a slight change in Definition 1: the tree labeled by a derivation in nuPLL must be over $\mathbb{N}^\omega$ instead of $\{1,2\}^*$, in order to deal with infinitely branching derivations.

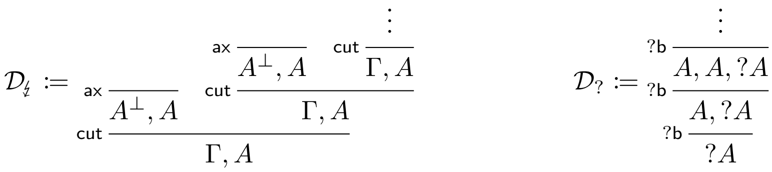
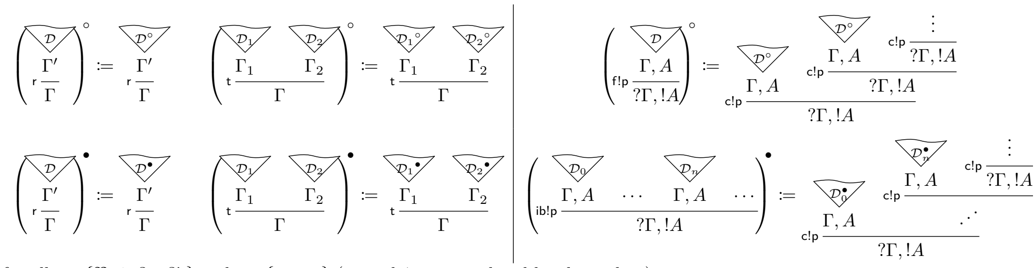


Figure 7 Two non-wellfounded and non-progressing coderivations in PLL^∞.



for all $\mathsf{r} \in \{\parr, \bot, ?\mathsf{w}, ?\mathsf{b}\}$ and $\mathsf{t} \in \{\mathsf{cut}, \otimes\}$ (ax and 1 are translated by themselves).

Figure 8 Translations $(\cdot)^\circ$ from PLL to PLL^∞, and $(\cdot)^\bullet$ from nuPLL to PLL^∞.

Furthermore, to achieve completeness for $\mathbf{P}/\mathsf{poly}$, these authors introduced second-order quantifiers and the co-absorption ($!\mathsf{b}$) and co-weakening ($!\mathsf{w}$) rules displayed in (1).

Cut-elimination steps for nuPLL are in Figures 3, 5, and 6. In particular, the step $\mathsf{ib!p}$-vs-$?\mathsf{b}$ in Figure 6 *pops* the first derivation $\mathcal{D}_0$ of $\mathsf{ib!p}$ out of the stream $(\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_n, \dots)$.

4.1 From infinitely branching proofs to non-wellfounded proofs

In this paper we explore a dual approach to the one of $\mathbf{nuPL}_{\forall\ell}$ (and nuPLL): instead of considering (wellfounded) derivations with infinite branching, we consider (non-wellfounded) coderivations with finite branching. For this purpose, the infinitary rule $\mathsf{ib!p}$ of nuPLL is replaced by the binary rule below, called **conditional promotion** ($\mathsf{c!p}$):

$$\mathsf{c!p}\,\frac{\Gamma, A \quad ?\Gamma, !A}{?\Gamma, !A} \tag{2}$$

▶ **Definition 7.** *We define the set of rules* $\mathsf{PLL}^\infty := \{\mathsf{ax}, \otimes, \parr, 1, \bot, \mathsf{cut}, ?\mathsf{b}, ?\mathsf{w}, \mathsf{c!p}\}$*. We also denote by* PLL^∞ *the set of coderivations over the rules in* PLL^∞*.*

In other words, PLL^∞ is the set of coderivations generated by the same rules as PLL, except that $\mathsf{f!p}$ is replaced by $\mathsf{c!p}$. From now on, we will only consider coderivations in PLL^∞.

▶ **Example 8.** Figure 7 shows two non-wellfounded coderivations in PLL^∞: $\mathcal{D}_\sharp$ (resp. $\mathcal{D}_?$) has an infinite branch of cut (resp. $?\mathsf{b}$) rules, and is (resp. is not) regular.

We can embed PLL and nuPLL into PLL^∞ via the conclusion-preserving translations $(\cdot)^\circ \colon \mathsf{PLL} \to \mathsf{PLL}^\infty$ and $(\cdot)^\bullet \colon \mathsf{nuPLL} \to \mathsf{PLL}^\infty$ defined in Figure 8 by induction on derivations: they map all rules to themselves except $\mathsf{f!p}$ and $\mathsf{ib!p}$, which are "unpacked" into non-wellfounded coderivations that iterate infinitely many times the rule $\mathsf{c!p}$.

An infinite chain of $\mathsf{c!p}$ rules (Figure 9) is a structure of interest in itself in PLL^∞.

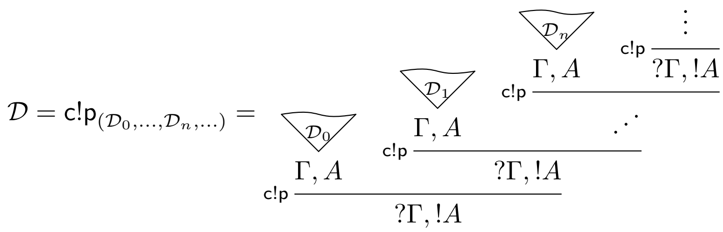
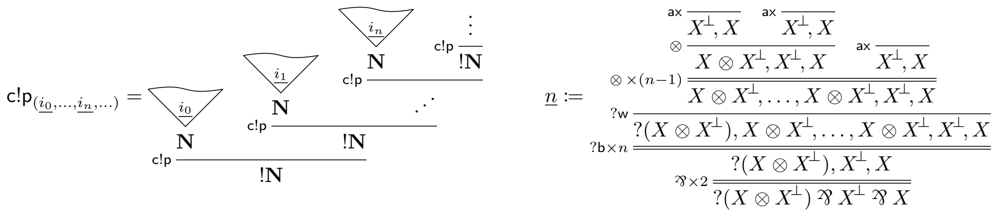


Figure 9 A non-wellfounded box in PLL^∞.

▶ **Definition 9.** *A* ***non-wellfounded box*** *(nwb for short) is a coderivation $\mathcal{D} \in \mathsf{PLL}^\infty$ with an infinite branch $\{\epsilon, 2, 22, \dots\}$ (the* ***main branch*** *of $\mathcal{D}$) all labeled by $\mathsf{c!p}$ rules as in Figure 9, where $!A$ in the conclusion is the* ***principal formula*** *of $\mathcal{D}$, and $\mathcal{D}_0, \mathcal{D}_1, \dots$ are the* ***calls*** *of $\mathcal{D}$. We denote $\mathcal{D}$ by $\mathsf{c!p}_{(\mathcal{D}_0,\dots,\mathcal{D}_n,\dots)}$.*

Let $\mathfrak{S} = \mathsf{c!p}_{(\mathcal{D}_0,\dots,\mathcal{D}_n,\dots)}$ be a nwb. We may write $\mathfrak{S}(i)$ to denote $\mathcal{D}_i$. We say that $\mathfrak{S}$ has ***finite support*** *(resp. is* ***periodic*** *with* ***period*** *k) if $\{\mathfrak{S}(i) \mid i \in \mathbb{N}\}$ is finite (resp. if $\mathfrak{S}(i) = \mathfrak{S}(k+i)$ for any $i \in \mathbb{N}$). A coderivation $\mathcal{D}$ has* ***finite support*** *(resp. is* ***periodic****) if any nwb in $\mathcal{D}$ has finite support (resp. is periodic).*

▶ **Example 10.** The only cut-free derivations of the formula $\mathbf{N} := !(X \multimap X) \multimap X \multimap X$ are of the form $\underline{n}$ below on the right, for all $n \in \mathbb{N}$, up to permutations of the rules $?\mathsf{w}$ and $\otimes$ (the derivations $\underline{0}$ and $\underline{1}$ in Example 4 are special cases of it)



(3)

Consider the nwb $\mathsf{c!p}_{(\underline{i_0},\dots,\underline{i_n},\dots)}$ above on the left, proving the formula $!\mathbf{N}$, where $i_j \in \{0,1\}$ for all $j \in \mathbb{N}$. Thus $\mathsf{c!p}_{(\underline{i_0},\dots,\underline{i_n},\dots)}$ has finite support, as its only calls can be $\underline{0}$ or $\underline{1}$, and it is periodic if and only if so is the infinite sequence $(i_0, \dots, i_n, \dots) \in \{0,1\}^\omega$.

The *cut-elimination* steps $\rightarrow_{\mathsf{cut}}$ for PLL^∞ are in Figures 3, 5, and 10. Computationally, they allow the $\mathsf{c!p}$ rule to be interpreted as a *coinductive* definition of a stream of type $!A$ from a stream of the same type to which an element of type A is prepended. In particular, the cut-elimination step $\mathsf{c!p}$ vs $?\mathsf{b}$ accesses the head of a stream: rule $?\mathsf{b}$ acts as a *pop* operator.

As a consequence, the nwb in Figure 9 constructs a stream $(\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_n, \dots)$ similarly to $\mathsf{ib!p}$ but, unlike the latter, all the $\mathcal{D}_i$'s may be pairwise distinct. The reader expert in linear logic can see a nwb as a box with possibly *infinitely many* distinct contents (its calls), while usual linear logic boxes (and $\mathsf{f!p}$ in PLL) provide infinitely many copies of the *same* content.

Rules $\mathsf{f!p}$ in PLL and $\mathsf{ib!p}$ in nuPLL are mapped by $(\cdot)^\circ$ and $(\cdot)^\bullet$ into nwbs, which are non-wellfounded coderivations. Hence, the cut-elimination steps $\mathsf{f!p}$ vs $\mathsf{f!p}$ in PLL and $\mathsf{ib!p}$ vs $\mathsf{ib!p}$ in nuPLL can only be simulated by infinitely many cut-elimination steps in PLL^∞.

Note that $\mathcal{D}_{\lightning} \in \mathsf{PLL}^\infty$ in Figure 7 is not cut-free, and if $\mathcal{D}_{\lightning} \rightarrow_{\mathsf{cut}} \mathcal{D}$ then $\mathcal{D} = \mathcal{D}_{\lightning}$: thus $\mathcal{D}_{\lightning}$ cannot reduce to a cut-free coderivation, and so the cut-elimination theorem fails in PLL^∞.

4.2 Consistency via a progressing criterion

In a non-wellfounded setting such as PLL^∞, any sequent is provable. Indeed, the (non-wellfounded) coderivation $\mathcal{D}_{\lightning}$ in Figure 7 shows that any non-empty sequent (in particular,

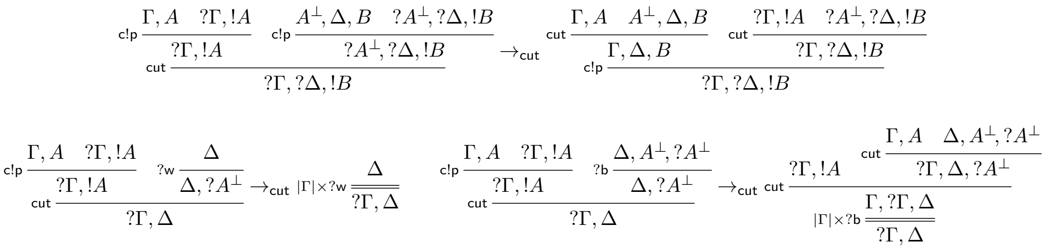


Figure 10 Exponential cut-elimination steps for coderivations of PLL^∞.

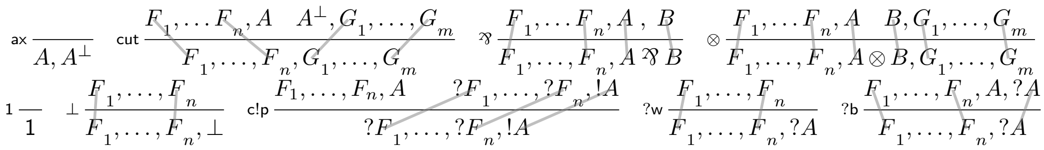


Figure 11 PLL^∞ rules: edges connect a formula in the conclusion with its parent(s) in a premise.

any formula) is provable in PLL^∞, and the empty sequent is provable in PLL^∞ by applying the cut rule on the conclusions B and $B^\perp$ (for any formula B) of two derivations $\mathcal{D}_\lightning$.

The standard way to recover logical consistency in non-wellfounded proof theory is to introduce a global soundness condition on coderivations, called *progressing criterion*. In PLL^∞, this criterion relies on tracking occurrences of !-formulas in a coderivation.

▶ **Definition 11.** *Let $\mathcal{D}$ be a coderivation in PLL^∞. It is* ***weakly progressing*** *if every infinite branch contains infinitely many right premises of* $\mathsf{c!p}$*-rules.*

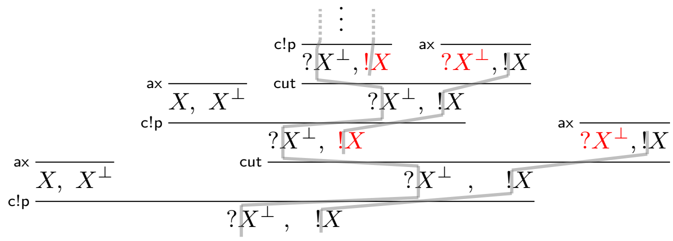
An occurrence of a formula in a premise of a rule r *is the* ***parent*** *of an occurrence of a formula in the conclusion if they are connected according to the edges depicted in Figure 11.*

A ***!-thread*** *(resp.* ***?-thread****) in $\mathcal{D}$ is a maximal sequence $(A_i)_{i\in I}$ of !-formulas (resp. ?-formulas) for some downward-closed $I \subseteq \mathbb{N}$ such that A_{i+1} is the parent of A_i for all $i \in I$. A !-thread $(A_i)_{i\in I}$ is* ***progressing*** *if A_j is in the conclusion of a* $\mathsf{c!p}$ *for infinitely many $j \in I$. $\mathcal{D}$ is* ***progressing*** *if every infinite branch contains a progressing !-thread. We define* pPLL^∞ *(resp.* wpPLL^∞*) as the set of progressing (resp. weak-progressing) coderivations in* PLL^∞.

▶ Remark 12. Clearly, any progressing coderivation is weakly progressing too, but the converse fails (Example 14), therefore $\mathsf{pPLL}^\infty \subsetneq \mathsf{wpPLL}^\infty$. Moreover, the main branch of any nwb contains by definition a progressing !-thread of its principal formula.

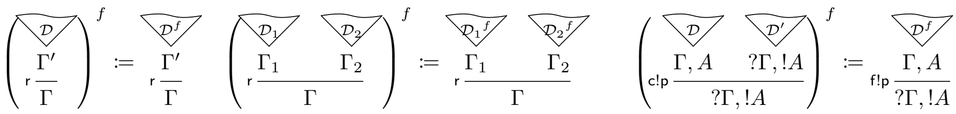
▶ Remark 13. Any branch $\mathcal{B}$ in a progressing coderivation $\mathcal{D}$ contains at most (and hence exactly) one progressing !-thread. This follows by maximality of !-threads and the fact that conclusions of $\mathsf{c!p}$ rules contain at most one !-formula. As a consequence, any infinite !-thread τ of a branch $\mathcal{B}$ in a progressing coderivation $\mathcal{D}$ must be progressing.

▶ **Example 14.** Coderivations in Figure 7 are not weakly progressing (hence, not progressing): the rightmost branch of $\mathcal{D}_\lightning$, i.e., the branch $\{\epsilon, 2, 22, \dots\}$, and the unique branch of $\mathcal{D}_?$ are infinite and contain no $\mathsf{c!p}$-rules. In contrast, the nwb $\mathsf{c!p}_{(\underline{i_0},\dots,\underline{i_n},\dots)}$ in Example 10 is progressing by Remark 12, since its main branch is the only infinite branch. Below, a regular, weakly progressing but not progressing coderivation ($!X$ in the conclusion of $\mathsf{c!p}$ is a cut formula, so the branch $\{\epsilon, 2, 21, 212, 2121, \dots\}$ is infinite but has no progressing !-thread).



▶ **Lemma 15.** *Let* Γ *be a sequent. Then,* $\vdash_{\mathsf{PLL}} \Gamma$ *if and only if* $\vdash_{\mathsf{wpPLL}^\infty} \Gamma$.

Proof. Given $\mathcal{D} \in \mathsf{PLL}$, $\mathcal{D}^\circ \in \mathsf{PLL}^\infty$ preserves the conclusion and is progressing, hence weakly progressing (see Remark 12). Conversely, given a weakly progressing coderivation $\mathcal{D}$, we define a derivation $\mathcal{D}^f \in \mathsf{PLL}$ with the same conclusion by applying, bottom-up, the translation:



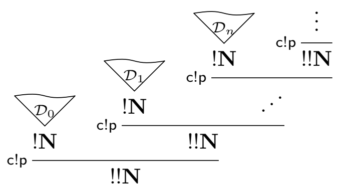
with $\mathsf{r} \neq \mathsf{c!p}$. Note that the derivation $\mathcal{D}^f$ is well-defined because $\mathcal{D}$ is weakly progressing. ◀

▶ **Corollary 16.** *The empty sequent is not provable in* wpPLL^∞ *(and hence in* pPLL^∞*).*

Proof. If the empty sequent were provable in wpPLL^∞, then there would be a cut-free derivation $\mathcal{D} \in \mathsf{PLL}$ of the empty sequent by Lemma 15 and Theorem 5, but this is impossible since cut is the only rule in PLL that could have the empty sequent in its conclusion. ◀

4.3 Recovering (weak forms of) regularity

The progressing criterion cannot capture the finiteness condition of the rule $\mathsf{ib!p}$ in the derivations in nuPLL. By means of example, consider the nwb below, which is progressing but cannot be the image of the rule $\mathsf{ib!p}$ via $(\cdot)^\bullet$ (see Figure 8) since $\{\mathcal{D}_i \mid i \in \mathbb{N}\}$ is infinite.



with $\mathcal{D}_i = \mathsf{c!p}_{(\underbrace{1,\dots,1}_{i},0,\dots)}$ for each $i \in \mathbb{N}$. (4)

To identify in pPLL^∞ the coderivations corresponding to derivations in nuPLL and in PLL via the translations $(\cdot)^\bullet$ and $(\cdot)^\circ$, respectively, we need additional conditions.

▶ **Definition 17.** *A coderivation is* ***weakly regular*** *if it has only finitely many distinct sub-coderivations whose conclusions are left premises of* $\mathsf{c!p}$*-rules; it is* ***finitely expandable*** *if any branch contains finitely many* cut *and* $?\mathsf{b}$ *rules. We denote by* wrPLL^∞ *(resp.* rPLL^∞*) the set of weakly regular (resp. regular) and finitely expandable coderivations in* pPLL^∞*.*

▶ Remark 18. Regularity implies weak regularity and the converse fails as shown in Example 19 below, so $\mathsf{rPLL}^\infty \subsetneq \mathsf{wrPLL}^\infty$. Given $\mathcal{D} \in \mathsf{PLL}^\infty$ progressing and finitely expandable, it is regular (resp. weakly regular) if and only if any nwb in $\mathcal{D}$ is periodic (resp. has finite support).

▶ **Example 19.** Coderivations $\mathcal{D}_{\lightning}$ and $\mathcal{D}_{?}$ in Figure 7 are not finitely expandable, as their infinite branch has infinitely many cut or ?b, but they are weakly regular, since they have no c!p rules. The coderivation in (4) is not weakly regular because $\{\mathcal{D}_i \mid i \in \mathbb{N}\}$ is infinite.

An example of a weakly regular but not regular coderivation is the nwb $\mathsf{c!p}_{(\underline{i_0},\dots,\underline{i_n},\dots)}$ in Example 10 when the infinite sequence $(i_j)_{j\in\mathbb{N}} \in \{0,1\}^\omega$ is not periodic: $\underline{0}$ and $\underline{1}$ are the only coderivations ending in the left premise of a c!p rule (so the nwb is weakly regular), but there are infinitely many distinct coderivations ending in the right premise of a c!p rule (so the nwb is not regular). Moreover, that nwb is finitely expandable, as it contains no ?b or cut.

The sets rPLL^∞ and wrPLL^∞ are the non-wellfounded counterparts of PLL and nuPLL, respectively. Indeed, we have the following correspondence via the translations $(\cdot)^\circ$ and $(\cdot)^\bullet$.

▶ **Proposition 20.** *The following statements hold.*

1. *If $\mathcal{D} \in \mathsf{PLL}$ (resp. $\mathcal{D} \in \mathsf{nuPLL}$) with conclusion Γ, then $\mathcal{D}^\circ \in \mathsf{rPLL}^\infty$ (resp. $\mathcal{D}^\bullet \in \mathsf{wrPLL}^\infty$) with conclusion Γ, and every c!p in $\mathcal{D}^\circ$ (resp. $\mathcal{D}^\bullet$) belongs to a nwb.*
2. *If $\mathcal{D}' \in \mathsf{rPLL}^\infty$ (resp. $\mathcal{D}' \in \mathsf{wrPLL}^\infty$) and every c!p in $\mathcal{D}'$ belongs to a nwb, then there is $\mathcal{D} \in \mathsf{PLL}$ (resp. $\mathcal{D} \in \mathsf{nuPLL}$) such that $\mathcal{D}^\circ = \mathcal{D}'$ (resp. $\mathcal{D}^\bullet = \mathcal{D}'$).*

1. By straightforward induction on $\mathcal{D} \in \mathsf{PLL}$ (resp. $\mathcal{D} \in \mathsf{nuPLL}$).
2. $\mathbf{d}(\mathcal{D}) \in \mathbb{N}$ by Lemma 45. We can then prove the statement by induction on $\mathbf{d}(\mathcal{D})$. We prove the statement by induction on the measure $\mathbf{d}(\mathcal{D})$ (see Definition 42 and Lemma 45).

Progressing and weak progressing coincide in finitely expandable coderivations.

▶ **Lemma 21.** *Let $\mathcal{D} \in \mathsf{PLL}^\infty$ be finitely expandable. If $\mathcal{D} \in \mathsf{wpPLL}^\infty$ then any infinite branch contains the main branch of a nwb. Moreover, $\mathcal{D} \in \mathsf{pPLL}^\infty$ if and only if $\mathcal{D} \in \mathsf{wpPLL}^\infty$.*

Proof. Let $\mathcal{D} \in \mathsf{wpPLL}^\infty$ be finitely expandable, and let $\mathcal{B}$ be an infinite branch in $\mathcal{D}$. By finite expandability there is $h \in \mathbb{N}$ such that $\mathcal{B}$ contains no conclusion of a cut or ?b with height greater than h. Moreover, by weakly progressing there is an infinite sequence $h \leq h_0 < h_1 < \dots < h_n < \dots$ such that the sequent of $\mathcal{B}$ at height h_i has shape $?\Gamma_i, !A_i$. By inspecting the rules in Figure 1, each such $?\Gamma_i, !A_i$ can be the conclusion of either a ?w or a c!p (with right premise $?\Gamma_i, !A_i$). So, there is a k large enough such that, for any $i \geq k$, only the latter case applies (and, in particular, $\Gamma_i = \Gamma$ and $A_i = A$ for some Γ, A). Therefore, h_k is the root of a nwb. This also shows $\mathcal{D} \in \mathsf{pPLL}^\infty$. By Remark 12, $\mathsf{pPLL}^\infty \subseteq \mathsf{wpPLL}^\infty$. ◀

By inspecting the steps in Figures 3, 5, and 10, we prove the following preservations.

▶ **Proposition 22.** *Cut elimination preserves weak-regularity, regularity and finite expandability. Therefore, if $\mathcal{D} \in \mathsf{X}$ with $\mathsf{X} \in \{\mathsf{rPLL}^\infty, \mathsf{wrPLL}^\infty\}$ and $\mathcal{D} \to_{\mathsf{cut}} \mathcal{D}'$, then also $\mathcal{D}' \in \mathsf{X}$.*

Proof. By inspection of the cut-elimination steps defined in Figures 3, 5, and 10. ◀

5 Continuous cut-elimination

Cut-elimination for (finitary) sequent calculi proceeds by introducing a proof rewriting strategy that stepwise decreases an appropriate termination ordering (see, e.g, [36]). Typically, these proof rewriting strategies consist on pushing upward the topmost cuts via the cut-elimination steps in order to eventually eliminate them.

A somewhat dual approach is investigated in the context of non-wellfounded proofs [5, 18]. It consists on *infinitary* proof rewriting strategies that gradually push upward the bottommost cuts. In this setting, the progressing condition is essential to guarantee *productivity*, i.e., that such proof rewriting strategies construct strictly increasing approximations of the cut-free proof, which can thus be obtained as a (well-defined) *limit*.

A major obstacle of this approach arises when the bottommost cut r is below another one r'. In this case, no cut-elimination step can be applied to r, so proof rewriting runs into an apparent stumbling block. To circumvent this problem, in [5, 18] a special cut-elimination step is introduced, which merges r and r' in a single, generalized cut rule called *multicut*.

In this section we study a continuous cut-elimination method that does not rely on multicut rules, following an alternative idea in which the notion of approximation plays an even more central rule, inspired by the topological approaches to infinite trees [8]. To this end, we assume the reader familiar with basic definitions on domain-theory (see, e.g., [3]).

5.1 Approximating coderivations

We introduce *open coderivations* to approximate coderivations. They form Scott-domains, on top of which we define *continuous cut elimination*. We also exploit them to *decompose* a finitely expandable and progressing coderivation into a *finite* approximation and a finite sequence of nwbs.

▶ **Definition 23.** *We define the set of rules* $\mathsf{oPLL}^\infty := \mathsf{PLL}^\infty \cup \{\mathsf{hyp}\}$*, where* $\mathsf{hyp} := {\scriptstyle\mathsf{hyp}}\frac{}{\Gamma}$ *for any sequent* Γ.[4] *We will also refer to* oPLL^∞ *as the set of coderivations over* oPLL^∞*, which we call* ***open coderivations****. An open coderivation is* ***normal*** *if no cut-elimination step can be applied to it, that is, if one premise of each* cut *is a* hyp*. An* ***open derivation*** *is a derivation in* oPLL^∞*. We denote by* $\mathsf{oPLL}^\infty(\Gamma)$ *the set of open coderivations with conclusion* Γ*.*

▶ **Definition 24.** *Let* $\mathcal{D}$ *be an open coderivation,* $\mathcal{V} \subseteq \{1,2\}^*$ *be a set of mutually incomparable (w.r.t. the prefix order) nodes of* $\mathcal{D}$*, and* $\{\mathcal{D}'_\nu\}_{\nu\in\mathcal{V}}$ *be a set of open coderivations where* $\mathcal{D}'_\nu$ *has the same conclusion as the subderivation* $\mathcal{D}_\nu$ *of* $\mathcal{D}$*. We denote by* $\mathcal{D}\{\mathcal{D}'_\nu/\nu\}_{\nu\in\mathcal{V}}$*, the open coderivation obtained by replacing each* $\mathcal{D}_\nu$ *with* $\mathcal{D}'_\nu$*. When* $\mathcal{V}$ *is finite, we will also use the alternative notation* $\mathcal{D}(\mathcal{D}'_{\nu_1}/\nu_1,\dots,\mathcal{D}'_{\nu_n}/\nu_n)$*.*

The ***pruning*** *of* $\mathcal{D}$ *over* $\mathcal{V}$ *is the open coderivation* $\lfloor\mathcal{D}\rfloor_\mathcal{V} = \mathcal{D}\{\mathsf{hyp}/\nu\}_{\nu\in\mathcal{V}}$*. If* $\mathcal{D}$ *and* $\mathcal{D}'$ *are open coderivations, we say that* $\mathcal{D}$ *is an* ***approximation*** *of* $\mathcal{D}'$ *(noted* $\mathcal{D} \preceq \mathcal{D}'$*) iff* $\mathcal{D} = \lfloor\mathcal{D}'\rfloor_\mathcal{V}$ *for some* $\mathcal{V} \subseteq \{1,2\}^*$*. An approximation is* ***finite*** *if it is an open derivation. We denote by* $\mathcal{K}(\mathcal{D})$ *(resp.* $\mathcal{K}^f(\mathcal{D})$*) the set of approximations (resp. finite approximations) of* $\mathcal{D}$*.*

Note that $\mathcal{D}$ and $\lfloor\mathcal{D}\rfloor_\mathcal{V}$ (and hence $\mathcal{D}'$ if $\mathcal{D} \preceq \mathcal{D}'$) have the same conclusion. Any open coderivation $\mathcal{D}$ is the supremum of its finite approximations, i.e., $\mathcal{D} = \bigsqcup_{\mathcal{D}'\in\mathcal{K}^f(\mathcal{D})} \mathcal{D}'$. Indeed:

▶ **Proposition 25.** *For any sequent* Γ*, the poset* $(\mathsf{oPLL}^\infty(\Gamma), \preceq)$ *is a Scott-domain with least element the open derivation* hyp *and with maximal elements the coderivations (in* PLL^∞*) with conclusion* Γ*. The compact elements are precisely the open derivations in* $\mathsf{oPLL}^\infty(\Gamma)$*.*

Cut-elimination steps essentially do not increase the size of open derivations, hence:

▶ **Lemma 26.** $\to_{\mathsf{cut}}$ *over open derivations is strongly normalizing and confluent.*

Proof. For $\mathcal{D}$ an open derivation, let $\mathsf{C}(\mathcal{D})$ be the number of $\mathsf{c!p}$ in $\mathcal{D}$ and $\mathsf{H}(\mathcal{D})$ be the sum of the sizes of all subderivations of $\mathcal{D}$ whose root is the conclusion of a cut rule. If $\mathcal{D} \to_{\mathsf{cut}} \mathcal{D}'$ via:

[4] Previously introduced notions and definitions on coderivations extend to open coderivations in the obvious way, e.g. the global conditions of Definitions 11 and 17 and the cut-elimination relation $\to_{\mathsf{cut}}$.

- a commutative cut-elimination step, then $\mathsf{C}(\mathcal{D}) = \mathsf{C}(\mathcal{D}')$, $|\mathcal{D}| = |\mathcal{D}'|$ and $\mathsf{H}(\mathcal{D}) > \mathsf{H}(\mathcal{D}')$;
- a multiplicative cut-elimination (Figure 3), then $\mathsf{C}(\mathcal{D}) = \mathsf{C}(\mathcal{D}')$ and $|\mathcal{D}| > |\mathcal{D}'|$;
- an exponential cut-elimination step (Figure 10), then $\mathsf{C}(\mathcal{D}) > \mathsf{C}(\mathcal{D}')$.

Since the lexicographic order over the triples $(\mathsf{C}(\mathcal{D}), |\mathcal{D}|, \mathsf{H}(\mathcal{D})) \in \omega^3$ is wellfounded, we conclude that there is no infinite sequence $(\mathcal{D}_i)_{i\in\mathbb{N}}$ such that $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_i \to_{\mathsf{cut}} \mathcal{D}_{i+1}$.

Finally, since cut-elimination $\to_{\mathsf{cut}}$ is strongly normalizing over open derivations and it is locally confluent by inspection of critical pairs, by Newman's lemma it is also confluent. ◀

Progressing and finitely expandable coderivations can be approximated in a canonical way. Indeed, by Lemma 21 we have:

▶ **Proposition 27.** *If $\mathcal{D} \in \mathsf{pPLL}^\infty$ is finitely expandable, then there is a prebar $\mathcal{V} \subseteq \{1,2\}^*$ of $\mathcal{D}$ such that each $v \in \mathcal{V}$ is the root of a* nwb *in $\mathcal{D}$.*

▶ **Definition 28.** *Let $\mathcal{D} \in \mathsf{pPLL}^\infty$ be finitely expandable. The* ***decomposition prebar*** *of $\mathcal{D}$ is the minimal prebar $\mathcal{V}$ of $\mathcal{D}$ such that, for all $\nu \in \mathcal{V}$, $\mathcal{D}_\nu$ is a* nwb*. We denote with* border$(\mathcal{D})$ *such a bar and we set* base$(\mathcal{D}) := \lfloor \mathcal{D} \rfloor_{\mathsf{border}(\mathcal{D})}$.

Note that, by weak König lemma, in the above definition border$(\mathcal{D})$ is finite and base$(\mathcal{D})$ is a finite approximation of $\mathcal{D}$.

5.2 Domain-theoretic approach to continuous cut-elimination

In this subsection we define *maximal and continuous infinitary cut-elimination strategies* (mc-ices), that is, specific rewriting strategies generating infinite reduction chains, whose limits are cut-free open coderivations. In other words, a mc-ices computes a (Scott-)continuous function from open coderivations to cut-free open coderivations. Then, we introduce the *height-by-height* mc-ices, a notable example of mc-ices that will be used for our results, and we show that any two mc-icess compute the same (Scott-)continuous function.

In what follows, σ denotes a countable sequence of coderivations, and $\sigma(i)$ denotes the $(i+1)$-th coderivation in σ. We denote the length of a sequence σ by $\mathsf{length}(\sigma) \leq \omega$.

▶ **Definition 29.** *An* ***infinitary cut-elimination strategy*** *(or* ices *for short) is a family $\sigma = \{\sigma_{\mathcal{D}}\}_{\mathcal{D}\in\mathsf{oPLL}^\infty}$ where, for all $\mathcal{D} \in \mathsf{oPLL}^\infty$, $\sigma_{\mathcal{D}}$ is a sequence of open coderivations such that $\sigma_{\mathcal{D}}(0) = \mathcal{D}$ and $\sigma_{\mathcal{D}}(i) \to_{\mathsf{cut}} \sigma_{\mathcal{D}}(i+1)$ for all $0 \leq i < \mathsf{length}(\sigma_{\mathcal{D}})$. Given an* ices *$\sigma$, we define the function $f_\sigma \colon \mathsf{oPLL}^\infty(\Gamma) \to \mathsf{oPLL}^\infty(\Gamma)$ as $f_\sigma(\mathcal{D}) := \bigsqcup_{i=0}^{\mathsf{length}(\sigma_{\mathcal{D}})} \mathsf{cf}(\sigma_{\mathcal{D}}(i))$ where $\mathsf{cf}(\mathcal{D}_i)$ is the greatest cut-free approximation of $\mathcal{D}_i$ (w.r.t. $\preceq$).*[5]

An ices *σ is a* mc-ices *if it is:*

- ***maximal****: $\sigma_{\mathcal{D}}(\mathsf{length}(\sigma_{\mathcal{D}}))$ is normal for any open derivation $\mathcal{D}$ ($\mathsf{length}(\sigma_{\mathcal{D}}) < \omega$ by Lemma 26);*
- ***(Scott)-continuous****: f_σ is Scott-continuous.*

Roughly, a maximal ices is an ices that applies cut-elimination steps to open derivations until a normal (possibly cut-free) open derivation is reached. Together with continuity, this allows us to define $f_\sigma(\mathcal{D})$ as the supremum of the normal open derivation obtained by applying the cut-elimination steps to the finite approximations of a coderivation $\mathcal{D}$,

[5] The function f_σ is well-defined since $\{\mathsf{cf}(\sigma_{\mathcal{D}}(i)) \mid i \geq 0\}$ is a direct set in $(\mathsf{oPLL}^\infty(\Gamma), \preceq)$, therefore its supremum exists by Proposition 25.

that is, $f_\sigma(\mathcal{D}) = \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} f_\sigma(\mathcal{D}') = \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} \mathsf{cf}(\sigma'_{\mathcal{D}}(\mathsf{length}(\mathcal{D}')))$, where $\mathsf{length}(\sigma_{\mathcal{D}}) < \omega$ by Lemma 26.

The following property states that all mc-icess induce the same continuous function, an easy consequence of Lemma 26 and continuity.

▶ **Proposition 30.** *If σ and σ' are two* mc-ices*s, then $f_\sigma = f_{\sigma'}$.*

Proof. For any open derivation $\mathcal{D}$, since σ and σ' are maximal, we have that $\sigma_{\mathcal{D}}(\mathsf{length}(\sigma_{\mathcal{D}}))$ and $\sigma'_{\mathcal{D}}(\mathsf{length}(\sigma'_{\mathcal{D}}))$ are normal, and so $\sigma_{\mathcal{D}}(\mathsf{length}(\sigma_{\mathcal{D}})) = \sigma'_{\mathcal{D}}(\mathsf{length}(\sigma'_{\mathcal{D}}))$ by Lemma 26. Hence:

$$f_\sigma(\mathcal{D}) = \mathsf{cf}(\sigma_{\mathcal{D}}(\mathsf{length}(\sigma_{\mathcal{D}}))) = \mathsf{cf}(\sigma'_{\mathcal{D}}(\mathsf{length}(\sigma'_{\mathcal{D}}))) = f_{\sigma'}(\mathcal{D})$$

Now, let $\mathcal{D}$ be an open coderivation. Since oPLL^∞ is a Scott-domain (Proposition 25), then $\mathcal{D} = \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} \mathcal{D}'$. We conclude that $f_\sigma(\mathcal{D}) = \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} f_\sigma(\mathcal{D}') = \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} f_{\sigma'}(\mathcal{D}') = f_{\sigma'}(\mathcal{D})$ because f_σ and $f_{\sigma'}$ are continuous. ◀

Therefore, we define a specific mc-ices we use in our proofs, where cut-elimination steps are applied in a deterministic way to the minimal reducible cut-rules.

▶ **Definition 31.** *The **height-by-height** ices is defined as $\sigma^\infty = \{\sigma^\infty_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ where $\sigma^\infty_{\mathcal{D}}(0) = \mathcal{D}$ for each $\mathcal{D} \in \mathsf{oPLL}^\infty$, and $\sigma^\infty_{\mathcal{D}}(i+1)$ is the open coderivation obtained by applying a cut-elimination step to the rightmost reducible* cut*-rule with minimal height in $\sigma^\infty_{\mathcal{D}}(i)$.*

▶ **Proposition 32.** *The* ices *σ^∞ is a* mc-ices*.*

Proof. By Lemma 26, any open derivation $\mathcal{D}$ normalizes in $n_{\mathcal{D}} \in \mathbb{N}$ steps; so, if $\mathcal{D}$ is an open derivation, $\mathsf{length}(\sigma^\infty_{\mathcal{D}}) = n_{\mathcal{D}}$ with $\sigma^\infty_{\mathcal{D}}(n_{\mathcal{D}})$ normal by definition of σ^∞. Hence, σ^∞ is maximal. To conclude we have to show that f_{σ^∞} is continuous. Since $\sigma^\infty_{\mathcal{D}}(i)$ is defined by applying a finite number of cut-eliminations steps to $\mathcal{D}$, then for each $i \in \mathbb{N}$ such that $\mathsf{length}(\sigma^\infty)$ there is $\mathcal{D}_i \in \mathcal{K}(\mathcal{D})$ such that $\mathsf{cf}(\sigma^\infty_{\mathcal{D}}(i)) \preceq f_{\sigma^\infty}(\mathcal{D}_i)$. Thus $f_{\sigma^\infty}(\mathcal{D}) \preceq \bigsqcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} f_{\sigma^\infty}(\mathcal{D}')$. We conclude since $\bigsqcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} f_{\sigma^\infty}(\mathcal{D}') \preceq f_{\sigma^\infty}(\mathcal{D})$ because σ^∞ is monotone by construction. ◀

▶ **Example 33.** For any finite approximation $\mathcal{D}$ of the (non-weakly progressing, non-finitely expandable) open coderivation $\mathcal{D}_{\not{\iota}}$, we have $f_{\sigma^\infty}(\mathcal{D}) = \mathsf{hyp}$, so $f_{\sigma^\infty}(\mathcal{D}_{\not{\iota}}) = \mathsf{hyp}$ by continuity.

5.3 Productivity of Continuous Cut-elimination

We conclude this section by proving continuous cut-elimination theorem, the main contribution of this paper, establishing a productivity result and showing that continuous cut-elimination preserves all global conditions. For this purpose, we introduce the notion of chain of cut-rules, which allows us to keep track of the dynamic of cut-elimination steps during infinitary rewriting. Note that the definition of cut-chain is the analogue of the *multi-cut reduction sequences* from [5].

▶ **Definition 34** (Chains)**.** *Let $\sigma = \{\sigma_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ be an* ices*. We write $\mathsf{r}_i \mapsto_\sigma \mathsf{r}_{i+1}$ if r_{i+1} is a* cut*-rule in $\sigma_{\mathcal{D}}(i+1)$ produced by applying a cut-elimination step to the* cut*-rule r_i in $\sigma_{\mathcal{D}}(i)$. A* cut*-**chain** in $\sigma_{\mathcal{D}}$ is a maximal sequence $(\mathsf{r}_i)_{i<\alpha}$ of* cut *rules with $\alpha \leq \mathsf{length}(\sigma_{\mathcal{D}})$, such that r_i a rule in $\sigma_{\mathcal{D}}(i)$, and either $\mathsf{r}_i = \mathsf{r}_{i+1}$ or $\mathsf{r}_i \mapsto_\sigma \mathsf{r}_{i+1}$. It is **stable** if (it is infinite and) there is a $k \geq 0$ such that $\mathsf{r}_i = \mathsf{r}_k$ for all $i \geq k$.*

▶ **Notation 35.** *Let $\mathcal{D}$ be a coderivation. The **starting point** of a thread τ in $\mathcal{D}$ is its first element. A **progressing point** of a $!$-thread $!\rho$ in $\mathcal{D}$ is a sequent which is conclusion of a* c!p*-rule with principal $!$-formula the one in $!\rho$.*

A cut*-rule is* ***reducible*** *if a cut-elimination step can be applied to it, that is, if one of its premises is not the conclusion of a* cut*-rule.*

We denote by $\sigma^{\infty}_{\tau}(i)$ *the thread of* $\sigma^{\infty}_{\mathcal{D}}(i)$ *obtained by applying* σ^{∞} *to* τ *(notice that, if it exists, it is unique).*

▶ **Lemma 36.** *Let* $\mathcal{D} \in \mathsf{pPLL}^{\infty}$*. If* $\mathcal{D}$ *is not cut-free, then it contains a reducible* cut*-rule.*

Proof. By definition, if no cut-rule in $\mathcal{D}$ is reducible, then both premises of each cut-rule in $\mathcal{D}$ are conclusion of cut-rules. Therefore, if $\mathcal{D}$ is not cut-free, then any rule above a cut-rule is a cut-rule. This implies the existence of an infinite branch containing only cut-rules, contradicting progressivity of $\mathcal{D}$. ◀

▶ **Lemma 37.** *Let* $\mathcal{D} \in \mathsf{pPLL}^{\infty}$ *and* $(\mathsf{r}_j)_{j\geq 0}$ *an infinite* cut*-chain in* $\sigma^{\infty}_{\mathcal{D}}$ *with* r_0 *in* $\mathcal{D}$*. Then:*

1. *If* $(\mathsf{r}_j)_{j\geq 0}$ *is not stable, then there is no* $i_0 \geq 0$ *such that, for every* $i \geq i_0$*, either* $\mathsf{r}_{i+1} = \mathsf{r}_i$ *or* $\mathsf{r}_i \mapsto_{\sigma} \mathsf{r}_{i+1}$ *is a commuting step or a* $\mathsf{c!p}$*-vs-*$\mathsf{?b}$ *step.*
2. $(\mathsf{r}_j)_{j\geq 0}$ *is not stable;*
3. *there is some* $i_0 \geq 0$ *and a* $!$*-thread* τ *of* $\sigma_{\mathcal{D}}(i_0)$ *such that, for every* $i \geq i_0$*:*
   - **a.** *the* $!$*-formula that is active in* r_i *is the starting point of* $\sigma^{\infty}_{\tau}(i)$*,*
   - **b.** *if* $\mathsf{r}_i \mapsto_{\sigma} \mathsf{r}_{i+1}$ *then* r_i *is a* $\mathsf{c!p}$*-vs-*$\mathsf{c!p}$*, a* $\mathsf{c!p}$*-vs-*$\mathsf{?b}$ *or a commuting step,*
   - **c.** *there are infinitely many* $\mathsf{r}_i \mapsto_{\sigma} \mathsf{r}_{i+1}$ *such that* r_i *is a* $\mathsf{c!p}$*-vs-*$\mathsf{c!p}$ *step.*

Proof. Concerning Item 1, suppose $(\mathsf{r}_j)_{j\geq 0}$ is not stable, and there is some $i_0 \geq 0$ such that, for every $i \geq i_0$, either $\mathsf{r}_{i+1} = \mathsf{r}_i$ or $\mathsf{r}_i \mapsto_{\sigma} \mathsf{r}_{i+1}$ is a commuting step or a $\mathsf{c!p}$-vs-$\mathsf{?b}$ step. We first show that the set I of $i \geq i_0$ such that $\mathsf{r}_i \mapsto_{\sigma} \mathsf{r}_{i+1}$ is a $\mathsf{c!p}$-vs-$\mathsf{?b}$ step is finite. To this end, we notice that the cut-elimination step $\mathsf{c!p}$-vs-$\mathsf{?b}$ erases a $\mathsf{?b}$ rule r and creates a finite sequence of $\mathsf{?b}$ rules in the same branch of r; no other cut-elimination step creates $\mathsf{?b}$ rules. This means that, if I were infinite, either a branch containing infinite $\mathsf{?b}$ rules (and only finitely many $\mathsf{c!p}$ rules) exists already in $\mathcal{D}$, or $\mathcal{D}$ contains a branch with infinitely many $\mathsf{c!p}$ rules whose principal $!$-formula will be eventually active for a cut of the form $\mathsf{c!p}$-vs-$\mathsf{?b}$. Both cases are impossible by progressivity of $\mathcal{D}$. Therefore, there is $i_1 \geq i_0$ such that, for every $i \geq i_1$, either $\mathsf{r}_{i+1} = \mathsf{r}_i$ or $\mathsf{r}_i \mapsto_{\sigma} \mathsf{r}_{i+1}$ is a commuting step. Moreover, since $(\mathsf{r}_j)_{j\geq 0}$ is not stable there must be infinite commuting steps. We apply a similar reasoning and appeal to progressivity of $\mathcal{D}$.

To prove Item 2, it suffices to show that there is $i \geq 0$ such that $\mathsf{r} = \mathsf{r}_0 \neq \mathsf{r}_i$. Suppose towards contradiction that $\mathsf{r}_i = \mathsf{r}$ for every $i \geq 0$. By Lemma 36, for every $i \geq 0$ there is a $\mathsf{r}'_i \neq \mathsf{r}_i$ that is reduced at the i-th step of cut-elimination. Let h_j and k_i be the heights of r_i and r'_i in $\sigma^{\infty}_{\mathcal{D}}(i)$ respectively. By definition of σ^{∞}, it must be that either $k_i < h_j$ or $k_i = h_i$ and r'_i is at the right of r (remember that σ^{∞} applies a cut-elimination step to the rightmost reducible cut with minimal height). Notice that there cannot be infinitely many i with $k_i < h_i$. Indeed, this would induce a cut-elimination sequence $\sigma^{\infty}_{\mathcal{D}'}$ with infinite length on a finite approximation $\mathcal{D}'$ of $\mathcal{D}$, contradicting Lemma 26. This means that there is $i_0 \geq 0$ such that $k_i = h_i = h_{i_0}$ for every $i \geq i_0$, and r'_i is at the right of r_i. We observe that there are only finitely many (reducible) cuts at height h_{i_0} in any $\sigma^{\infty}_{\mathcal{D}}(i)$, and moreover any cut-elimination step applied to a cut with height h_{i_0} produces at most one cut rule with the same height. We conclude that infinitely many r'_i belong to the same cut-chain, which is therefore not stable. Furthermore, the only cut reduced by this cut-chain are $\parr$-vs-$\otimes$ and $\mathsf{c!p}$-vs-$\mathsf{?b}$, since these are the only cut-elimination steps that reduce a cut producing a new one with the same height. However, since $\parr$-vs-$\otimes$-steps shrink the size of the cut-formulas, there must be infinitely many $\mathsf{c!p}$-vs-$\mathsf{?b}$-steps in the cut-chain. This contradicts Item 1.

Let us finally show Item 3. Note that since $(\mathsf{r}_j)_{j\geq 0}$ is infinite, no ax-vs-cut or ?w-vs-c!p cut-elimination step is ever applied to r_j, otherwise the chain would be finite. Also, there are finitely many $i \in \mathbb{N}$ such that the active formulas of r_j are $⅋$- or $\otimes$-formulas, because each $\otimes$-vs-$⅋$ cut-elimination step reduces the size of the cut-formula. Notice also that there are finitely many ?b-vs-c!p steps $\mathsf{r}_i \mapsto \mathsf{r}_{i+1}$ such that r_{i+1} is the topmost cut produced because, again, this cut-elimination step decreases the size of the formula. Therefore, there exists $i_0 \geq 0$ and a !-thread τ of $\sigma_{\mathcal{D}}(i_0)$ such that, for every $i \geq i_0$ the !-formula that is active in r_i is the starting point of $\sigma^{\infty}_{\tau}(i)$. This shows Item 3a. Moreover, Item 3b follows by the fact that any r_i with $i \geq i_0$ has exponential active formulas and the fact that $\mathsf{r}_i \mapsto \mathsf{r}_{i+1}$ is neither ax-vs-cut not ?w-vs-c!p. By Item 1 and Item 2, this implies Item 3c. ◀

We introduce some terminology to trace cut-rules during cut-elimination. The names yard, clewline and buntline are borrowed from the sailing jargon. The yard is a horizontal spar on which a square sail is attached.[6] The clewlines and buntlines are lines attached respectively to the corner and to the foot of a square sail, which is furled up on the yard.[7] The intuition is that, during cut-elimination, a yard eventually moves upwards together with the end-points of the buntlines, which are attached to the foot of the sail. Meanwhile the clewlines are pulled down-ward. We then use the existence of such a clewline for each cut-chain to ensure the existence of a progressing thread in each infinite branch created by eliminating the cut-rules of an infinite cut-chain.

▶ **Definition 38.** *A **yard** is an infinite* cut*-chain $(\mathsf{r}_i)_{i<\alpha}$ such that if $\mathsf{r}_{i+1} \neq \mathsf{r}_i$, then r_i is the $\leq_{\mathcal{T}}$-minimal right-most* cut*-rule produced by the cut-elimination step applied to r_i.*

*A thread τ is a **buntline** of a yard $(\mathsf{r}_i)_{i\geq 0}$ if there is a $i_0 \in \mathbb{N}$ such that the starting point of $\sigma^{\infty}_{\tau}(i)$ is the active formula of r_i for all $i \geq i_0$.*

*A !-thread $!\rho$ is a **clewline** of a yard $(\mathsf{r}_i)_{i\geq 0}$ if there is a $i_0 \in \mathbb{N}$ such that:*

- *the starting point of $\sigma^{\infty}_{!\rho}(i)$ is not the active formula of a* cut*-rule in $\sigma^{\infty}_{\mathcal{D}}(i)$ for any $i \geq 0$;*
- *the conclusion of r_i contains a formula in $\sigma^{\infty}_{!\rho}(i)$ for any $i \geq i_0$;*
- *For any $k > 0$ there is a $i_k \geq i_0$ such that $\sigma^{\infty}_{!\rho}(i_k)$ has at least k points right below r_{i_k} in $\sigma^{\infty}_{\mathcal{D}}(i_k)$.*

▶ **Lemma 39** (Furl up the sail!)**.** *Let $\mathcal{D} \in \mathsf{pPLL}^{\infty}$. If $(\mathsf{r}_j)_{j\geq 0}$ is a yard in $\sigma^{\infty}_{\mathcal{D}}$, then it has a (unique) clewline $!\rho$.*

Proof. By Lemma 37.3a there is a buntline τ of the yard $(\mathsf{r}_j)_{j\geq 0}$. Moreover, by definition and Lemma 37.3b and Lemma 37.3c, there is a $i_0 \geq 0$ such that:

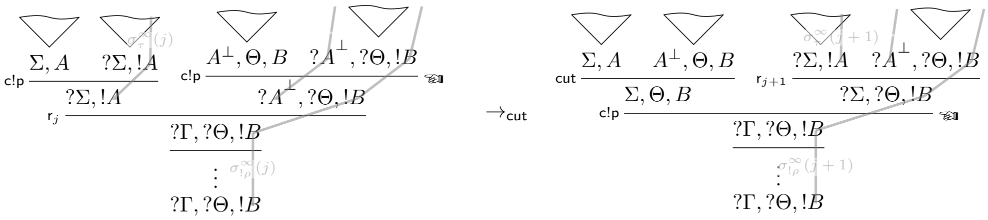
- r_{i_0} is of the form c!p-vs-c!p,
- for every $i \geq i_0$, if $\mathsf{r}_i \mapsto_{\sigma} \mathsf{r}_{i+1}$ then r_i is a c!p-vs-c!p, a c!p-vs-?b or a commuting step.

Moreover, it must be that one of the c!p rules right above r_{i_0} has principal !-formula active for r_{i_0}, and the other c!p rule has principal !-formula that appears both in the premise and in the conclusion of r_{i_0}. Let $!\rho$ be the !-thread containing the latter !-formula (it is unique by Remark 13). It is easy to check that, for all $i \geq i_0$:

(i) the starting point of $\sigma^{\infty}_{!\rho}(i)$ is not the active formula of a cut-rule in $\sigma^{\infty}_{\mathcal{D}}(i)$,
(ii) the conclusion of r_i contains a formula in $\sigma^{\infty}_{!\rho}(i)$.

[6] See https://en.wikipedia.org/wiki/Yard_(sailing).
[7] See https://en.wikipedia.org/wiki/Clewlines_and_buntlines.

This follows by the fact that every cut in a yard has minimal height and the fact that every $\mathsf{r}_i \mapsto_\sigma \mathsf{r}_{i+1}$ with $i \geq i_0$ is a c!p-vs-c!p, a c!p-vs-?b or a commuting step. To conclude that $!\rho$ is a clewline we observe that by Lemma 37.3c there are infinitely many $j \geq 0$ such that $\mathsf{r}_j \mapsto_\sigma \mathsf{r}_{j+1}$ is a c!p-vs-c!p step. By Items (i) and (ii) the principal !-formula of one of the two c!p rules right above r_j has a progressing point of $!\rho$. Then, the sub-coderivation of $\sigma^\infty_{\mathcal{D}}(j)$ containing $\sigma^\infty_{!\rho}(j)$ and the sub-coderivation of $\sigma^\infty_{\mathcal{D}}(j+1)$ containing $\sigma^\infty_{!\rho}(j+1)$ are of the following form, where we mark with ☜ the progressing points of $!\rho$ in focus.



This shows that there are infinitely many j such that $\sigma^\infty_{!\rho}(j+1)$ has strictly more progressing points right below r_{j+1} then $\sigma^\infty_{!\rho}(j)$ has below r_j. Moreover, we observe that the number of progressing points of $!\rho$ that are below cuts in the yard cannot decrease. Therefore, $!\rho$ is a clewline. ◀

We now have all technical tools requires to prove the so-called *productivity* result for f_{σ^∞}, that is, that if $\mathcal{D}$ is a progressing coderivation, then $f_{\sigma^\infty}(\mathcal{D})$ is a (well-defined) hyp-free and cut-free coderivation. In addition, we show that the same reasoning we use to ensure productivity also ensures that the progressing condition is preserved by f_{σ^∞}.

▶ **Theorem 40.** *If $\mathcal{D} \in \mathsf{pPLL}^\infty$, then so is $f_{\sigma^\infty}(\mathcal{D})$.*

Proof. We have to prove that if $\mathcal{D} \in \mathsf{pPLL}^\infty$, then $f_{\sigma^\infty}(\mathcal{D})$ is a well-defined cut-free coderivation, that is, $f_{\sigma^\infty}(\mathcal{D})$ contains no hyp-rule and any infinite branch in $f_{\sigma^\infty}(\mathcal{D})$ contains a progressing !-thread.

Let $\mathcal{B}$ be a branch of $f_{\sigma^\infty}(\mathcal{D})$. If $\mathcal{B}$ is also in $\sigma^\infty_{\mathcal{D}}(k)$ for a $k \in \mathbb{N}$ (therefore in all $\sigma^\infty_{\mathcal{D}}(j)$ with $j \geq k$), then it must be hyp-free by definition of the cut-elimination steps and, in case it is infinite, it has a progressing !-thread by Proposition 22.

Otherwise there is no $j \in \mathbb{N}$ such that $\mathcal{B}$ is a branch of $\sigma^\infty_{\mathcal{D}}(j)$. In this case,

- either there is an infinite cut-chain $(\mathsf{r}_j)_{j\geq 0}$ in $\sigma^\infty_{\mathcal{D}}$ and a $j_0 \in \mathbb{N}$ such that the branch $\mathcal{B}_j$ of $\sigma^\infty_{\mathcal{D}}(j)$ containing the conclusion of r_j is an initial segment of $\mathcal{B}$ for all $j \geq j_0$. Note that this cut-chain is a yard by construction, and that the sequence $(\mathcal{B}_j)_{j\geq 0}$ is well-ordered with supremum $\mathcal{B}$ by definition of σ^∞.
  In this case, by Lemma 39 there is clewline for $(\mathsf{r}_j)_{j\geq 0}$ and there is a j_0 such that for all $j \geq j_0$ if $\mathsf{r}_j \neq \mathsf{r}_{j+1}$, then $\mathcal{B}_{j+1}$ strictly contains $\mathcal{B}_j$ and the number of progressing points of the clewline below r_{j+1} increases. We conclude that $!\rho$ is the progressing !-thread of $\mathcal{B}$, and that $\mathcal{B}$ is hyp-free by construction because it is infinite;
- or we can define a sequence $(\mathsf{r}_j)_{j\geq 0}$ of cut-rules such that r_j is the bottom-most cut-rule in the initial segment $\mathcal{B}_j$ of $\mathcal{B}$ in $\sigma^\infty_{\mathcal{D}}(j)$. As in the previous case we have that the sequence $(\mathcal{B}_j)_{j\geq 0}$ is well-ordered with supremum $\mathcal{B}$ by definition of σ^∞.
  In this case $(\mathsf{r}_j)_{j\geq 0}$ is the union of finite cut-chains, that is, we have that $(\mathsf{r}_j)_{j\geq 0} = \prod_{k_i}(\mathsf{r}_{k_i}, \dots, \mathsf{r}_{(k_{i+1}-1)})$ for some $k_i \in \mathbb{N}$ with $0 = k_0$ and $k_i < k_{i+1}$ for all $i \in \mathbb{N}$. Thus we have that r_{j+1} is either obtained by applying a cut-elimination step to r_j, or that $j+1 = k_i$ if a cut-elimination step applied to r_j introduces no new cut-rules. By progressiveness of

$\mathcal{D}$, we know that the infinite branch of $\mathcal{D}$ containing the conclusion of all r_{k_i} contains a progressing thread τ. Then, since any cut-chain in $(\mathsf{r}_j)_{j\geq 0}$ is finite, there must be some $j_0 \geq 0$ such that the starting point of $\sigma^{\infty}_{\tau}(j)$ is not the active formula of r_j for any $j \geq j_0$. This means that $\mathcal{B}$ is progressing and hyp-free.

◀

▶ Remark 41. In our setting, yards play the same role of multicuts (see, e.g., [5, 14, 17, 1, 2]). To be more precise, the same role of the bottom-most cut-rule in a pile of cut-rules is represented by the multicut. Note that, at the pure syntactical level, if we consider a multicut as a rule and we define multicut-elimination steps, then the number of the premises of such a rule may increase indefinitively during cut-elimination. In fact, the derivation $\mathcal{D}_{\not{\imath}}$ in Figure 7 would reduce, via multicut-elimination, to a (progressing) derivation made of a multicut with infinite premises conclusion of ax-rules.

$$\dfrac{\dfrac{}{A^{\perp}, A}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{A^{\perp}, A}\,\mathsf{ax} \qquad \dfrac{\vdots}{\Gamma, A}\,\mathsf{cut}}{\Gamma, A}\,\mathsf{cut}}{\Gamma, A}\,\mathsf{cut} \qquad \to^{*}_{\mathsf{cut}} \qquad \dfrac{\dfrac{}{A, A^{\perp}}\,\mathsf{ax} \quad \cdots \quad \dfrac{}{A, A^{\perp}}\,\mathsf{ax} \quad \cdots}{A}\,\mathsf{multicut}$$

The most interesting aspect of our technique is that it allows us to focus on the study of the dynamics of the bottom-most cut-rule in a pile of cut-rules (corresponding to a multicut) to deduce important properties of the limit coderivations. This is possible because in progressive derivations there cannot be infinite piles of cut-rules, therefore a cut-elimination step is eventually applied to a rule on top of a pile of cut-rules, which therefore is progressively pushed by the height-by-height mc-ices below the pile, or simply removed by a cut-elimination step involving one of the cut-rules of the pile. This interesting dynamic is hidden by multicut-elimination, which lacks the granularity required to observe it.

5.4 Continuous Cut-elimination preserves (weak-)regularity

To prove that continuous cut-elimination preserves regularity and weak-regularity, we define the notion of *depth* of a coderivation, akin in linear logic, as the maximal number of nested nwbs.

▶ **Definition 42.** *Let $\mathcal{D} \in \mathsf{PLL}^{\infty}$. The **nesting level of a sequent occurrence** Γ in $\mathcal{D}$ is the number $\mathbf{nl}_{\mathcal{D}}(\Gamma)$ of nodes below it that are the root of a call of a* nwb. *The **nesting level of a formula (occurrence)** A in $\mathcal{D}$, noted $\mathbf{nl}_{\mathcal{D}}(A)$, is the nesting level of the sequent that contain that formula. The **nesting level of a rule** r in $\mathcal{D}$, noted $\mathbf{nl}_{\mathcal{D}}(\mathsf{r})$ (resp. **of a sub-coderivation** $\mathcal{D}'$ of $\mathcal{D}$, noted $\mathbf{nl}_{\mathcal{D}}(\mathcal{D}')$), is the nesting level of the conclusion of* r *(resp. conclusion of $\mathcal{D}'$).*

*The **depth of** $\mathcal{D}$ is* $\mathbf{d}(\mathcal{D}) \coloneqq \sup_{\mathsf{r}\in\mathcal{D}}\{\mathbf{nl}_{\mathcal{D}}(\mathsf{r})\} \in \mathbb{N} \cup \{\infty\}$.

▶ Remark 43. All calls of a nwb have the same nesting level. Moreover, each of the sequents of its main branch have nesting level 0.

Cut-elimination $\to_{\mathsf{cut}}$ on PLL^{∞} enjoys the following property.

▶ **Lemma 44.** *Let $\mathcal{D}, \mathcal{D}' \in \mathsf{PLL}^{\infty}$. If $\mathcal{D} \to_{\mathsf{cut}} \mathcal{D}'$ then $\mathbf{d}(\mathcal{D}) \geq \mathbf{d}(\mathcal{D}')$.*

Proof. By inspection of the cut-elimination steps in Figures 3, 5, and 10. ◀

▶ **Lemma 45.** *If $\mathcal{D}$ is weakly regular then $\mathbf{d}(\mathcal{D}) \in \mathbb{N}$.*

Proof. Since $\mathcal{D}$ is weakly regular, it has only finitely many distinct sub-coderivations whose conclusion is the left premise of a $\mathsf{c!p}$ rule. Therefore, $\mathbf{d}(\mathcal{D}) \in \mathbb{N}$. ◀

▶ **Proposition 46.** *If $\mathcal{D}$ is progressing and finitely expandable then so is $f_{\sigma^\infty}(\mathcal{D})$.*

Proof. By Theorem 40, $f_{\sigma^\infty}(\mathcal{D})$ is cut-free and progressing. To conclude, we need to show that any infinite branch $\mathcal{B}$ of $f_{\sigma^\infty}(\mathcal{D})$ has only finitely many $?\mathsf{b}$ rules. To this end, we notice that the cut-elimination step $\mathsf{c!p}$-vs-$?\mathsf{b}$ erases a $?\mathsf{b}$ rule r and creates a finite sequence of $?\mathsf{b}$ rules in the same branch of r. No other cut-elimination step creates $?\mathsf{b}$ rules. This means that, if $\mathcal{B}$ had infinitely many $?\mathsf{b}$ rules, either a branch containing infinite $?\mathsf{b}$ rules exists already in $\mathcal{D}$, or $\mathcal{D}$ contains a branch with infinitely many $\mathsf{c!p}$ rules whose principal !-formula will be eventually active for a cut of the form $\mathsf{c!p}$-vs-$?\mathsf{b}$. Both cases are impossible by progressiveness of $\mathcal{D}$. ◀

▶ **Proposition 47.** *Let $\mathcal{D} \in \mathsf{wrPLL}^\infty$ (resp. rPLL^∞). Then $f_{\sigma^\infty}(\mathcal{D})$ admits a decomposition prebar, and $\mathsf{base}(f_{\sigma^\infty}(\mathcal{D})) = \mathsf{base}(\sigma^\infty_{\mathcal{D}}(n))$ for some $n \geq 0$.*

Proof. By Theorem 40 and Proposition 46, $f_{\sigma^\infty}(\mathcal{D})$ is a hyp-free, cut-free, progressing and finitely expandable coderivation. By Proposition 27 $f_{\sigma^\infty}(\mathcal{D})$ admits a decomposition prebar $\mathsf{border}(\mathcal{D}) = \{v_1, \ldots, v_k\}$. By continuity of f_{σ^∞}, this means that there is $n \geq 0$ such that $\mathsf{base}(\sigma^\infty_{\mathcal{D}}(n)) = \mathsf{base}(f_{\sigma^\infty}(\mathcal{D}))$. Note that $\mathsf{base}(\sigma^\infty_{\mathcal{D}}(n))$ exists by Propositions 22 and 27. ◀

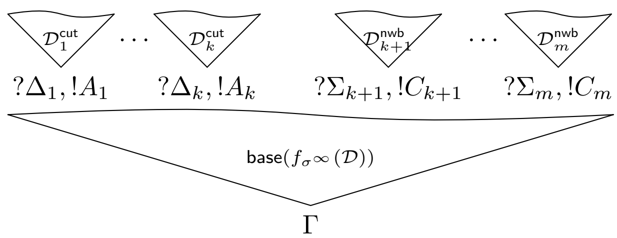
▶ **Theorem 48.** *Continuous cut-elimination preserves (weak-)regularity. That is, if $\mathcal{D} \in \mathsf{wrPLL}^\infty$ (resp. $\mathcal{D} \in \mathsf{rPLL}^\infty$), then so is $f_{\sigma^\infty}(\mathcal{D})$.*

Proof. To prove the statement we define a maximal and *transfinite* ices $\sigma = \{\sigma_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ eliminating the cut and preserving (weak) regularity, finite expandability and (weak) progressivity. Then show that we can define a mc-ices $\sigma^* = \{\sigma^*_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ by "compressing" σ, using a technique similar to the one in [32]. This allows us to conclude since, by Proposition 30, we have that $f_{\sigma^\infty} = f_{\sigma^*}$. In particular, every transfinite ices that we will construct will satisfy the following conditions for any limit ordinal $\lambda \leq \mathsf{length}(\sigma_{\mathcal{D}})$:

(I) $\sigma^\infty_{\mathcal{D}}(\lambda) = \bigsqcup_{i<\lambda} \tilde{\mathcal{D}}_i$ for some $\tilde{\mathcal{D}}_i$ finite approximations of $\sigma^\infty_{\mathcal{D}}(i)$,
(II) if h_i is the height of the cut-rule reduced at the i-th step of $\sigma^\infty_{\mathcal{D}}$, then $\lim_{i<\lambda}(h_i) = \infty$.

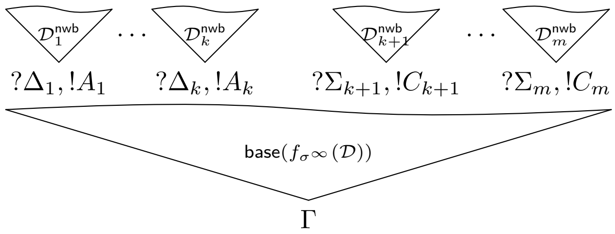
By Lemma 45 we know that $d = \mathbf{d}(\mathcal{D})$ is finite. Then we define $\sigma_{\mathcal{D}}$ as follows:

- If $d = 0$, then by Proposition 27 $\mathcal{D}$ is a derivation and there is a maximal (finite) cut-elimination sequence $\sigma_{\mathcal{D}}$ rewriting $\mathcal{D}$ to a cut-free and hyp-free derivation (see Lemma 26).
- If $d > 0$, then we know by Proposition 47 that there is $n_{\mathsf{base}(\mathcal{D})} \geq 0$ and a finite sequence σ^{base} such that $\sigma^{\mathsf{base}}(0) = \mathcal{D}$ and $\mathsf{base}(f_{\sigma^\infty}(\mathcal{D})) = \mathsf{base}(\sigma^{\mathsf{base}}(n_{\mathsf{base}(\mathcal{D})}))$ where

$\sigma^{\mathsf{base}}(n_{\mathsf{base}(\mathcal{D})}) =$  (5)

where $\mathcal{D}_i^{\mathsf{cut}}$ is made of a cut-rule with premises two nwb*s* $\mathcal{D}'_i$ and $\mathcal{D}''_i$ for each $i \in \{1, \ldots, k\}$ and $\mathcal{D}_i^{\mathsf{nwb}}$ is a nwb for each $i \in \{k+1, \ldots, m\}$.
We let σ' be the sequence of cut-elimination steps of length $\omega + 1$ that reduces only the cuts c!p-vs-c!p in the coderivations $\mathcal{D}_i^{\mathsf{cut}}$ with $i \in \{1, \ldots, k\}$ in such a way that $\sigma'(\omega)$ has the following shape:

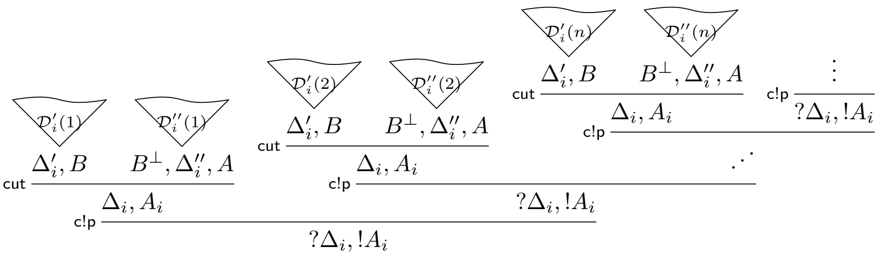
$\sigma'(\omega) =$



(6)

where for $i \in \{1, \ldots, k\}$ $\mathcal{D}_i^{\mathsf{nwb}}$ is the nwb whose calls are the coderivations obtained by cutting each j-th call $\mathcal{D}'_i(j)$ of the nwb $\mathcal{D}'_i$ with the corresponding j-th call $\mathcal{D}''_i(j)$ of the nwb $\mathcal{D}''_i$. That is,

$\mathcal{D}_i^{\mathsf{nwb}} =$



Note that, since $\mathcal{D}$ is (weakly) regular, then each $\mathcal{D}_i^{\mathsf{nwb}}$ is also (weakly) regular. By induction hypothesis, each call of the nwbs $\mathcal{D}_i^{\mathsf{nwb}}$ with $i \in \{1, \ldots, m\}$ has strictly smaller depth. Therefore for all $i \in \{1, \ldots, m\}$ we can define a cut-elimination sequence σ_i^{nwb} such that $\sigma_i^{\mathsf{nwb}}(0) = \mathcal{D}_i^{\mathsf{nwb}}$ that satisfies conditions (I) and (II), eliminates the cut and preserves (weak) regularity, finite expandability and (weak) progressiveness.
We define the sequence $\sigma_{\mathcal{D}}$ with transfinite length which first performs all cut-elimination steps required to remove reducible cut-rule in $\lfloor \mathcal{D} \rfloor_{\mathsf{border}(\mathcal{D})}$, then performs all c!p-vs-c!p steps, and finally applies cut-elimination steps to each call of the resulting new nwb*s* created by merging the calls. That is,

$$\sigma_{\mathcal{D}} = \sigma^{\mathsf{base}} \cdot \sigma' \cdot \prod_{i=1}^{m} \sigma_i^{\mathsf{nwb}}$$

where the product of sequences denotes their concatenation. By construction $\sigma_{\mathcal{D}}$ satisfies conditions (I) and (II), eliminates the cut and preserves (weak) regularity, finite expandability and (weak) progressiveness.

It is clear that $\sigma := \{\sigma_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ is a maximal transfinite ices. To conclude we have to define a mc-ices $\sigma^* = \{\sigma^*_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ containing only sequences of length at most ω such that $f_{\sigma^*}(\mathcal{D})$ is the same coderivation of the limit of $\sigma_{\mathcal{D}}$. We construct each $\sigma^*_{\mathcal{D}}$ by induction on $d = \mathbf{d}(\mathcal{D})$.

- If $d = 0$, then $\sigma_{\mathcal{D}}$ is finite and $\sigma^*_{\mathcal{D}} \coloneqq \sigma_{\mathcal{D}}$.

- Otherwise $d > 0$. By induction hypothesis, for all $i \in \{1, \dots, m\}$, we can construct from each σ_i^{nwb} a sequence σ_i^* with $\mathsf{length}(\sigma_i^*) \leq \omega$ such that $\sigma_i^*(0) = \mathcal{D}_i^{\mathsf{nwb}}$ and having the same limit, i.e., $\sigma_i^*(\mathsf{length}(\sigma_i^*)) = \hat{\sigma}_{\mathcal{D}}(\mathsf{length}(\sigma_i^{\mathsf{nwb}}))$. This means that the following cut-elimination sequence
$$\hat{\sigma}_{\mathcal{D}} \coloneqq \sigma^{\mathsf{base}} \cdot \sigma' \cdot \prod_{i=1}^{m} \sigma_i^*$$
has the same limit as $\sigma_{\mathcal{D}}$ and satisfies conditions (I) and (II). We notice that any cut-elimination step in σ_i^* commutes with any cut-elimination step in σ_j^* for any $i, j \in \{1, \dots, m\}$. Moreover, any cut-elimination step in σ_i^* with $i \in \{k+1, \dots, m\}$ commutes with any cut-elimination step in σ'. Finally, for any cut-elimination step in σ_i^* with $i \in \{1, \dots, k\}$ there is a $j_0 \geq 0$ such that it commutes with the j-th cut-elimination step in σ' for any $j \geq j_0$. This means that we can define $\sigma_{\mathcal{D}}^*$ by reordering the cut-elimination steps from $\hat{\sigma}_{\mathcal{D}}$ in such a way that we alternate one step in σ' with m steps from each σ_i^*. Clearly, $\sigma_{\mathcal{D}}^*$ is such that $\mathsf{length}(\sigma_{\mathcal{D}}^*) \leq \omega$, and it satisfies conditions (I) and (II). Moreover, since $\hat{\sigma}_{\mathcal{D}}$ satisfies condition (II), it is easy to show that this reordering preserves the limit, i.e., $\sigma_{\mathcal{D}}^*(\mathsf{length}(\sigma_{\mathcal{D}}^*)) = \hat{\sigma}_{\mathcal{D}}(\mathsf{length}(\hat{\sigma}_{\mathcal{D}}))$. Finally, since $\sigma_{\mathcal{D}}^*$ satisfies condition (I), the ices $\sigma^* \coloneqq \{\sigma_{\mathcal{D}}^*\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ is a (maximal and) continuous ices, i.e., f_{σ^*} is Scott-continuous. Therefore, σ^* is a $\mathsf{mc\text{-}ices}$ and, by Proposition 30, we have:
$$\hat{\sigma}_{\mathcal{D}}(\mathsf{length}(\hat{\sigma}_{\mathcal{D}})) = f_{\sigma^*}(\mathcal{D}) = f_{\sigma^\infty}(\mathcal{D})$$
which implies that $f_{\sigma^\infty}(\mathcal{D})$ is (weakly) regular, finitely expandable and (weakly) progressive.

◀

6 Relational semantics for non-wellfounded proofs

Here we define a denotational model for oPLL^∞ based on *relational semantics*, which interprets an open coderivation as the union of the interpretations of its finite approximations, as in [17]. We show that relational semantics is sound for oPLL^∞, but not for its extension with digging.

Relational semantics interprets exponential by finite multisets, denoted by brackets, e.g., $[x_1, \dots, x_n]$; $+$ denotes the *multiset union*, and $\mathcal{M}_f(X)$ denotes the set of finite multisets over a set X. To correctly define the semantics of a coderivation, we need to see sequents as *finite sequences* of formulas (taking their order into account), which means that we have to add an *exchange* rule to oPLL^∞ to swap the order of two consecutive formulas in a sequent.

▶ **Definition 49.** *We associate with each formula A a* ***set*** *$[\![A]\!]$ defined as follows:*
$$[\![X]\!] \coloneqq D_X \quad [\![1]\!] \coloneqq \{*\} \quad [\![A \otimes B]\!] \coloneqq [\![A]\!] \times [\![B]\!] \quad [\![!A]\!] \coloneqq \mathcal{M}_f([\![A]\!]) \quad [\![A^\perp]\!] \coloneqq [\![A]\!]$$
where D_X is an arbitrary set. For a sequent $\Gamma = A_1, \dots, A_n$, we set $[\![\Gamma]\!] \coloneqq [\![A_1 \parr \cdots \parr A_n]\!]$.

Given $\mathcal{D} \in \mathsf{PLL} \cup \mathsf{oPLL}^\infty$ with conclusion Γ, we set $[\![\mathcal{D}]\!] \coloneqq \bigcup_{n \geq 0} [\![\mathcal{D}]\!]_n \subseteq [\![\Gamma]\!]$, where $[\![\mathcal{D}]\!]_0 = \varnothing$ and, for all $i \in \mathbb{N} \setminus \{0\}$, $[\![\mathcal{D}]\!]_i$ is defined inductively according to Figure 12.

▶ **Example 50.** For the coderivations $\mathcal{D}_{\lightning}$ and $\mathcal{D}_?$ in Figure 7, $[\![\mathcal{D}_{\lightning}]\!] = [\![\mathcal{D}_?]\!] = \varnothing$. For the derivations $\underline{0}$ and $\underline{1}$ in Figure 2, $[\![\underline{0}]\!] = \{([\,], (x, x)) \mid x \in D_X\}$ and $[\![\underline{1}]\!] = \{([(x, y)], (x, y)) \mid x, y \in D_X\}$. For the coderivation $\mathsf{c!p}_{(\underline{i_0}, \dots, \underline{i_n}, \dots)}$ in Example 10 (with $i_j \in \{0, 1\}$ for all $j \in \mathbb{N}$), $[\![\mathsf{c!p}_{(\underline{i_0}, \dots, \underline{i_n}, \dots)}]\!] = \{[\,]\} \cup \Big\{[x_{i_0}, \dots, x_{i_n}] \in \mathcal{M}_f([\![\mathbf{N}]\!]) \mid n \in \mathbb{N},\ x_{i_j} \in [\![\underline{i_j}]\!]\ \forall 0 \leq j \leq n\Big\}$. For the

$$\left\llbracket \mathsf{ax}\,\frac{}{A, A^{\perp}} \right\rrbracket_n = \{\, (x, x) \mid x \in \llbracket A \rrbracket \,\} \qquad \left\llbracket \mathsf{cut}\,\frac{\overset{\mathcal{D}'}{\Gamma, A} \quad \overset{\mathcal{D}''}{\Delta, A^{\perp}}}{\Gamma, \Delta} \right\rrbracket_n = \left\{ (\vec{x}, \vec{y}) \;\middle|\; \exists z \in \llbracket A \rrbracket \text{ s.t. } \begin{array}{c} (\vec{x}, z) \in \llbracket \mathcal{D}' \rrbracket_{n-1} \\ \text{and} \\ (z, \vec{y}) \in \llbracket \mathcal{D}'' \rrbracket_{n-1} \end{array} \right\}$$

$$\left\llbracket \perp\,\frac{\overset{\mathcal{D}'}{\Gamma}}{\Gamma, \perp} \right\rrbracket_n = \{\, (\vec{x}, *) \mid \vec{x} \in \llbracket \mathcal{D}' \rrbracket_{n-1} \,\} \qquad \left\llbracket \parr\,\frac{\overset{\mathcal{D}'}{\Gamma, A, B}}{\Gamma, A \parr B} \right\rrbracket_n = \{\, (\vec{x}, (y, z)) \mid (\vec{x}, y, z) \in \llbracket \mathcal{D}' \rrbracket_{n-1} \,\}$$

$$\left\llbracket 1\,\frac{}{1} \right\rrbracket_n = \{*\} \qquad \left\llbracket \otimes\,\frac{\overset{\mathcal{D}'}{\Gamma, A} \quad \overset{\mathcal{D}''}{\Delta, B}}{\Gamma, \Delta, A \otimes B} \right\rrbracket_n = \left\{ (\vec{x}, \vec{y}, (x, y)) \;\middle|\; \begin{array}{c} (\vec{x}, x) \in \llbracket \mathcal{D}' \rrbracket_{n-1} \\ \text{and} \\ (\vec{y}, y) \in \llbracket \mathcal{D}'' \rrbracket_{n-1} \end{array} \right\} \qquad \left\llbracket \mathsf{hyp}\,\frac{}{\Gamma} \right\rrbracket_n = \varnothing$$

$$\left\llbracket ?\mathsf{w}\,\frac{\overset{\mathcal{D}'}{\Gamma}}{\Gamma, ?A} \right\rrbracket_n = \{\, (\vec{x}, [\,]) \mid \vec{x} \in \llbracket \mathcal{D}' \rrbracket_{n-1} \,\} \qquad \left\llbracket ?\mathsf{b}\,\frac{\overset{\mathcal{D}'}{\Gamma, A, ?A}}{\Gamma, ?A} \right\rrbracket_n = \{\, (\vec{x}, [y] + \mu) \mid (\vec{x}, y, \mu) \in \llbracket \mathcal{D}' \rrbracket_{n-1} \,\}$$

$$\left\llbracket \mathsf{c!p}\,\frac{\overset{\mathcal{D}'}{\Gamma, A} \quad \overset{\mathcal{D}''}{?\Gamma, !A}}{?\Gamma, !A} \right\rrbracket_n = \left\{ (\vec{[\,]}, [\,]) \right\} \cup \left\{ ([x_1] + \mu_1, \dots, [x_k] + \mu_k, [x] + \mu) \;\middle|\; \begin{array}{c} (x_1, \dots, x_k, x) \in \llbracket \mathcal{D}' \rrbracket_{n-1} \\ \text{and} \\ (\mu_1, \dots, \mu_k, \mu) \in \llbracket \mathcal{D}'' \rrbracket_{n-1} \end{array} \right\}$$

Figure 12 Inductive definition of the set $\llbracket \mathcal{D} \rrbracket_n$, for $n > 0$.

derivation $\underline{n}$ in Example 10 (for any $n \in \mathbb{N}$), $\llbracket \underline{n} \rrbracket = \{([(x_1, x_2), \dots, (x_n, x_{n+1})], (x_1, x_{n+1})) \mid x_1, \dots, x_{n+1} \in D_X\}$. Note that $\llbracket \underline{n} \rrbracket \cap \llbracket \underline{m} \rrbracket = \varnothing$ for all $n, m \in \mathbb{N}$ such that $n \neq m$, and that $\llbracket n \rrbracket$ is stable under permutations of the rules ?w and $\otimes$ in $\underline{n}$ (that is, if $\mathcal{D}$ is obtained from $\underline{n}$ by permuting the rules ?w or $\otimes$, then $\llbracket \mathcal{D} \rrbracket = \llbracket \underline{n} \rrbracket$).

By inspecting the cut-elimination steps and by continuity, we can prove the soundness of relational semantics with respect to cut-elimination (Theorem 56), thanks to the fact the interpretation of a coderivation is the union the interpretations of its finite approximation.

▶ **Lemma 51.** *Let $\mathcal{D}, \mathcal{D}' \in \mathsf{oPLL}^{\infty}$.*

1. *If $n \in \mathbb{N}$ and $\mathcal{D} \preceq \mathcal{D}'$, then $\llbracket \mathcal{D} \rrbracket_n \subseteq \llbracket \mathcal{D}' \rrbracket_n$.*
2. *If $\mathcal{D} \preceq \mathcal{D}'$ then $\llbracket \mathcal{D} \rrbracket \subseteq \llbracket \mathcal{D}' \rrbracket$.*

Proof. 1. By induction on $n \in \mathbb{N}$. By definition, $\llbracket \mathcal{D} \rrbracket_0 = \varnothing = \llbracket \mathcal{D}' \rrbracket_0$.
Let $n > 0$. Let us consider the bottommost rule r in $\mathcal{D}'$. As $\mathcal{D} \preceq \mathcal{D}'$, there are two cases:

- the bottommost rule in $\mathcal{D}$ is hyp and then $\llbracket \mathcal{D} \rrbracket_n = \varnothing \subseteq \llbracket \mathcal{D}' \rrbracket_n$;
- the bottommost rule in $\mathcal{D}$ is also r; thus, if r has no premises then $\mathcal{D} = \mathcal{D}'$ and hence $\llbracket \mathcal{D} \rrbracket_n = \llbracket \mathcal{D}' \rrbracket_n$; otherwise $\mathcal{D}_i \preceq \mathcal{D}'_i$ for every respective premise $\mathcal{D}_i, \mathcal{D}'_i$ of r in $\mathcal{D}, \mathcal{D}'$, and thus $\llbracket \mathcal{D} \rrbracket_n \subseteq \llbracket \mathcal{D}' \rrbracket_n$ easily follows from the inductive hypothesis, since $\llbracket \mathcal{D} \rrbracket_n$ and $\llbracket \mathcal{D}' \rrbracket_n$ only depend on $\llbracket \mathcal{D}_i \rrbracket_{n-1}$ and $\llbracket \mathcal{D}'_i \rrbracket_{n-1}$, respectively (see Figure 12).

2. According to Lemma 51.1, for all $n \in \mathbb{N}$, $\llbracket \mathcal{D} \rrbracket_n \subseteq \llbracket \mathcal{D}' \rrbracket_n \subseteq \bigcup_{n \in \mathbb{N}} \llbracket \mathcal{D}' \rrbracket_n = \llbracket \mathcal{D}' \rrbracket$. By minimality of the union, $\llbracket \mathcal{D} \rrbracket = \bigcup_{n \in \mathbb{N}} \llbracket \mathcal{D} \rrbracket_n \subseteq \llbracket \mathcal{D}' \rrbracket$. ◀

▶ **Definition 52.** *Let $\mathcal{T}$ be a tree. The **height** of a branch of $\mathcal{T}$ is the supremum (in $\mathbb{N} \cup \{\infty\}$ with the expected order) of the heights of its nodes.*

*A prebar $\mathcal{V}$ of a open coderivation labeling a tree $\mathcal{T}$ has **uniform height** $n \in \mathbb{N}$ if all and only the branches of $\mathcal{T}$ with height $> n$ have a node in $\mathcal{V}$, and all nodes in $\mathcal{V}$ have height n.*

▶ **Lemma 53.** *Let $\mathcal{D} \in \mathsf{oPLL}^{\infty}$. For every $n \in \mathbb{N}$ there is $\mathcal{D}_n \in \mathcal{K}^f(\mathcal{D})$ such that $\llbracket \mathcal{D}_n \rrbracket = \llbracket \mathcal{D} \rrbracket_n$.*

Proof. By induction on $n \in \mathbb{N}$. By definition, $[\![\mathcal{D}]\!]_0 = \varnothing$. Let $\mathcal{D}_0$ the open derivation made only of the rule hyp with the same conclusion as $\mathcal{D}$: then, $\mathcal{D}_0 \in \mathcal{K}^f(\mathcal{D})$ and $[\![\mathcal{D}_0]\!] = \varnothing = [\![\mathcal{D}]\!]_0$ (since $[\![\mathcal{D}_0]\!]_n = \varnothing$ for all $n \in \mathbb{N}$).

Let $n > 0$. Let $\mathcal{D}_n = \lfloor \mathcal{D} \rfloor_{\mathcal{V}_n}$ where $\mathcal{V}_n$ is a prebar of $\mathcal{D}$ with uniform height n. Thus, it is easy to check that $[\![\mathcal{D}_n]\!]_i = [\![\mathcal{D}]\!]_i$ for every $0 \leq i \leq n$, and $[\![\mathcal{D}_n]\!]_i = [\![\mathcal{D}]\!]_n$ for all $i \geq n$, since $[\![\mathcal{D}']\!]_0 = \varnothing$ for all $\mathcal{D}' \in \mathsf{oPLL}^\infty$ and $[\![\mathsf{hyp}]\!]_i = \varnothing$ for all $i \in \mathbb{N}$. ◀

▶ **Lemma 54.** *Let $\mathcal{D} \in \mathsf{oPLL}^\infty$. Then, $[\![\mathcal{D}]\!] = [\![\bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} \mathcal{D}']\!] = \bigcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} [\![\mathcal{D}']\!]$.*

Proof. By Proposition 25, $\mathcal{D} = \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} \mathcal{D}'$.

For the left-to-right inclusion, observe that for every $n \in \mathbb{N}$ there is $\mathcal{D}'_n \in \mathcal{K}^f(\mathcal{D})$ such that $[\![\bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} \mathcal{D}']\!]_n = [\![\mathcal{D}'_n]\!] \subseteq \bigcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} [\![\mathcal{D}']\!]$. Therefore, by minimality of the union,

$$[\![\mathcal{D}]\!] = \bigcup_{n \in \mathbb{N}} [\![\mathcal{D}]\!]_n = \bigcup_{n \in \mathbb{N}} [\![\bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} \mathcal{D}']\!]_n \subseteq \bigcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} [\![\mathcal{D}']\!].$$

As for the converse inclusion, we have that $\mathcal{D}' \preceq \mathcal{D}''$ implies $[\![\mathcal{D}']\!] \subseteq [\![\mathcal{D}'']\!]$. Hence, for all $\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})$, since $\mathcal{D}' \preceq \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} \mathcal{D}' = \mathcal{D}$, we have $[\![\mathcal{D}']\!] \subseteq [\![\mathcal{D}]\!]$. By minimality of the union, $\bigcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} [\![\mathcal{D}']\!] \subseteq [\![\mathcal{D}]\!]$. ◀

▶ **Lemma 55.** *Let σ is a $\mathsf{mc\text{-}ices}$, and $\mathcal{D} \in \mathsf{oPLL}^\infty(\Gamma)$. If $\mathcal{D}'' \in \mathcal{K}^f(f_\sigma(\mathcal{D}))$ then there is $\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})$ such that $\mathcal{D}'' \preceq f_\sigma(\mathcal{D}')$.*

Proof. As $\mathcal{D}'' \in \mathcal{K}^f(f_\sigma(\mathcal{D}))$, in particular $\mathcal{D}'' \preceq f_\sigma(\mathcal{D})$. Since $(\mathsf{oPLL}^\infty(\Gamma), \preceq)$ is a Scott-domain (Proposition 25), $\mathcal{D} = \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} \mathcal{D}'$ and hence, by continuity of f_σ (as σ is a $\mathsf{mc\text{-}ices}$), $f_\sigma(\mathcal{D}) = f_\sigma(\bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} \mathcal{D}') = \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} f_\sigma(\mathcal{D}')$. By compactness of $\mathcal{D}''$ (Proposition 25), from $\mathcal{D}'' \preceq f_\sigma(\mathcal{D}) = \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} f_\sigma(\mathcal{D}')$ it follows that $\mathcal{D}'' \preceq f_\sigma(\mathcal{D}')$ for some $\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})$. ◀

▶ **Theorem 56** (Soundness). *theoremsoundness*

1. *Let $\mathcal{D} \in \mathsf{oPLL}^\infty$. If $\mathcal{D} \to_{\mathsf{cut}} \mathcal{D}'$, then $[\![\mathcal{D}]\!] = [\![\mathcal{D}']\!]$.*
2. *Let $\mathcal{D} \in \mathsf{oPLL}^\infty$. If σ is a $\mathsf{mc\text{-}ices}$, then $[\![\mathcal{D}]\!] = [\![f_\sigma(\mathcal{D})]\!]$.*

Proof. 1. By straightforward inspection of the cut-elimination steps for oPLL^∞.

2. By definition of $\mathsf{mc\text{-}ices}$, for any $\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})$ we have $\mathcal{D}' \to^*_{\mathsf{cut}} f_\sigma(\mathcal{D}')$, so $[\![\mathcal{D}']\!] = [\![f_\sigma(\mathcal{D}')]\!]$ by Theorem 56.1. By Proposition 25, $\mathcal{D} = \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} \mathcal{D}'$. By continuity of f_σ, we have $f_\sigma(\mathcal{D}) = \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} f_\sigma(\mathcal{D}')$. Thus, using Lemma 54 in the first equality below,

$$[\![\mathcal{D}]\!] = \textstyle\bigcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} [\![\mathcal{D}']\!] = \bigcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} [\![f_\sigma(\mathcal{D}')]\!] = [\![\bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} f_\sigma(\mathcal{D}')]\!] = [\![f_\sigma(\mathcal{D})]\!]$$

where the third equality holds for the following reasons:

- $\subseteq$: for all $\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})$, $f_\sigma(\mathcal{D}') \preceq \bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} f_\sigma(\mathcal{D}')$ and so $[\![f_\sigma(\mathcal{D}')]\!] \subseteq [\![\bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} f_\sigma(\mathcal{D}')]\!]$ by Lemma 51; by minimality of the union, $\bigcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} [\![f_\sigma(\mathcal{D}')]\!] \subseteq [\![\bigsqcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} f_\sigma(\mathcal{D}')]\!]$;
- $\supseteq$: according to Lemma 55, for all $\mathcal{D}'' \in \mathcal{K}^f(f_\sigma(\mathcal{D}))$, there is $\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})$ such that $\mathcal{D}'' \preceq f_\sigma(\mathcal{D}')$, and hence $[\![\mathcal{D}'']\!] \subseteq [\![f_\sigma(\mathcal{D}')]\!] \subseteq \bigcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} [\![f_\sigma(\mathcal{D}')]\!]$ by Lemma 51; by Lemma 54 and minimality of the union, $[\![f_\sigma(\mathcal{D})]\!] = \bigcup_{\mathcal{D}'' \in \mathcal{K}^f(f_\sigma(\mathcal{D}))} [\![\mathcal{D}'']\!] \subseteq \bigcup_{\mathcal{D}' \in \mathcal{K}^f(\mathcal{D})} [\![f_\sigma(\mathcal{D}')]\!]$. ◀

By Theorem 56 and since cut-free coderivations have non-empty semantics, we have:

▶ **Corollary 57.** *Let $\mathcal{D} \in \mathsf{wpPLL}^\infty$. Then $[\![\mathcal{D}]\!] \neq \varnothing$.*

$$
{\scriptstyle ??\mathsf{d}}\,\frac{\Gamma, ??A}{\Gamma, ?A}
\qquad
\left[\!\!\left[\begin{array}{c}\mathcal{D}'\\ \hline {\scriptstyle ??\mathsf{d}}\,\dfrac{\Gamma, ??A}{\Gamma, ?A}\end{array} \right]\!\!\right]_0 = \varnothing
\qquad
\left[\!\!\left[\begin{array}{c}\mathcal{D}'\\ \hline {\scriptstyle ??\mathsf{d}}\,\dfrac{\Gamma, ??A}{\Gamma, ?A}\end{array} \right]\!\!\right]_n = \left\{ \left(\vec{x}, \sum_{i=1}^{m} \mu_i\right) \;\middle|\; (\vec{x}, [\mu_1, \dots, \mu_m]) \in [\![\mathcal{D}']\!]_{n-1},\ m \in \mathbb{N} \right\}
$$

Figure 13 The rule ??d and its interpretation in the relational semantics ($n > 0$).

Proof. If $\mathcal{D} \in \mathsf{wpPLL}^\infty$ is a cut-free coderivation, then weak-progressing ensures the existence of a bar $\mathcal{V}$ containing conclusions of rules in $\{\mathsf{ax}, 1, \mathsf{c!p}\}$. By weak König's lemma, $\lfloor \mathcal{D} \rfloor_\mathcal{V}$ is finite. Then, we prove by induction on $\lfloor \mathcal{D} \rfloor_\mathcal{V}$ that there is $n \geq 0$ such that $[\![\lfloor \mathcal{D} \rfloor_\mathcal{V}]\!]_n \neq \varnothing$, so that we conclude $\varnothing \neq [\![\lfloor \mathcal{D} \rfloor_\mathcal{V}]\!]_n \subseteq [\![\mathcal{D}]\!]_n \subseteq [\![\mathcal{D}]\!]$. As for the base case, notice that the interpretation of any coderivation ending with the $\mathsf{c!p}$ contains the element $(\vec{[\,]}, [\,])$, so it is never empty. The inductive steps are straightforward.

If $\mathcal{D}$ contains cut-rules, then $[\![\mathcal{D}]\!] = [\![f_\sigma(\mathcal{D})]\!]$ by Theorem 56. Since $f_\sigma(\mathcal{D})$ is cut-free, we conclude $[\![\mathcal{D}]\!] \neq \varnothing$ by the above reasoning. ◀

We define the set of rules $\mathsf{MELL}^\infty := \mathsf{PLL}^\infty \cup \{??\mathsf{d}\}$ where the rule $??\mathsf{d}$ (**digging**) is defined in Figure 13. We also denote by MELL^∞ the set of coderivations over the rules in MELL^∞. Relational semantics is naturally extended to MELL^∞ as shown in Figure 13.

The proof system MELL^∞ can be seen as a non-wellfounded version of MELL. We show that, as opposed to several fragments of PLL^∞, in any good fragment of MELL^∞ with digging, cut-elimination cannot reduce to cut-free coderivations *and* preserve both the progressing condition and relational semantics.

▶ **Theorem 58.** *Let $\mathsf{X} \subseteq \mathsf{MELL}^\infty$ contain non-wellfounded coderivations with $??\mathsf{d}$. Let $\to_{\mathsf{cut}+}$ be a cut-elimination relation on X preserving the progressing condition, containing $\to_{\mathsf{cut}}$ in Figures 3, 5, and 10 and reducing every coderivation in X to a cut-free one. Then, $\to_{\mathsf{cut}+}$ does not preserve relational semantics.*

Proof. Consider the coderivations $\mathcal{D}_{??\mathsf{d}}$ and $\widehat{\mathcal{D}_{??\mathsf{d}}}$ below, where $\mathcal{D} = \mathsf{c!p}_{(\underline{0},\underline{1},\underline{0},\underline{1},\dots)}$ and, for all $i \in \mathbb{N}$, $\mathcal{D}_i \in \{\mathsf{c!p}_{(\underline{k^i_0},\dots,\underline{k^i_n},\dots)} \mid k^i_j \in \mathbb{N} \text{ for all } j \in \mathbb{N}\}$ ($\underline{n}$ is defined in Example 10 for all $n \in \mathbb{N}$).

$$
\mathcal{D}_{??\mathsf{d}} := \dfrac{\begin{array}{c}\mathcal{D}\\ !\mathbf{N}\end{array} \qquad {\scriptstyle ??\mathsf{d}}\dfrac{{\scriptstyle \mathsf{ax}}\dfrac{}{??\mathbf{N}^\perp, !!\mathbf{N}}}{?\mathbf{N}^\perp, !!\mathbf{N}}}{!!\mathbf{N}}\,{\scriptstyle \mathsf{cut}}
\qquad\qquad
\widehat{\mathcal{D}_{??\mathsf{d}}} := {\scriptstyle \mathsf{c!p}}\dfrac{\begin{array}{c}\mathcal{D}_0\\ !\mathbf{N}\end{array} \qquad {\scriptstyle \mathsf{c!p}}\dfrac{\begin{array}{c}\mathcal{D}_1\\ !\mathbf{N}\end{array} \qquad \begin{array}{c}{\scriptstyle \mathsf{c!p}}\dfrac{\begin{array}{c}\mathcal{D}_n\\ !\mathbf{N}\end{array} \qquad {\scriptstyle \mathsf{c!p}}\dfrac{\vdots}{!!\mathbf{N}}}{} \\ \iddots \end{array}}{!!\mathbf{N}}}{!!\mathbf{N}}
$$

Coderivations $\widehat{\mathcal{D}_{??\mathsf{d}}}$ are the only cut-free and progressing ones with conclusion $!!\mathbf{N}$. Indeed, any cut-free coderivation of $!!\mathbf{N}$ or $!\mathbf{N}$ must end with a $\mathsf{c!p}$, and the only cut-free and progressing coderivations of $\mathbf{N}$ are the derivations of the form $\underline{n}$ for any $n \in \mathbb{N}$, up to permutations of the rules $?\mathsf{w}$ and $\otimes$ (other cut-free coderivations of $\mathbf{N}$ exist, but they have an infinite branch containing infinitely many $?\mathsf{b}$ rules and no $\mathsf{c!p}$ rules, hence they are not progressing). Therefore, for whatever definition of the cut-elimination steps concerning $??\mathsf{d}$ that preserves the progressing condition, necessarily $\mathcal{D}_{??\mathsf{d}}$ will reduce to $\widehat{\mathcal{D}_{??\mathsf{d}}}$, since $\mathcal{D}_{??\mathsf{d}}$ is progressing.

We show that $[\![\widehat{\mathcal{D}_{??\mathsf{d}}}]\!] \not\subseteq [\![\mathcal{D}_{??\mathsf{d}}]\!]$. First, it can be easily shown that if, in one of the $\mathcal{D}_i = \mathsf{c!p}_{(\underline{k^i_0},\dots,\underline{k^i_n},\dots)}$ in $\widehat{\mathcal{D}_{??\mathsf{d}}}$, one of the k^i_j is different from 0 or 1, then there is $x \in [\![\widehat{\mathcal{D}_{??\mathsf{d}}}]\!] \smallsetminus [\![\mathcal{D}_{??\mathsf{d}}]\!]$ (this basically follows from the fact that $[\![\underline{n}]\!] \cap [\![\underline{m}]\!] = \varnothing$ for all $n, m \in \mathbb{N}$ such that $n \neq m$, see Example 50). Let us now suppose that in $\widehat{\mathcal{D}_{??\mathsf{d}}}$, for all $i \in \mathbb{N}$, $\mathcal{D}_i = \mathsf{c!p}_{(\underline{k^i_0},\dots,\underline{k^i_n},\dots)}$ with $k^i_j \in \{0, 1\}$ for all $j \in \mathbb{N}$. Let $\hat{0}$ and $\hat{1}$ be any element of $[\![\underline{0}]\!]$ and $[\![\underline{1}]\!]$, respectively (see

Example 50). Note that $\hat{0} \neq \hat{1}$. It is easy to verify that $[[\hat{0}], [\hat{0}]], [[\hat{1}], [\hat{1}]] \notin [\![\mathcal{D}_{??\mathsf{d}}]\!]$, since $[\hat{0}, \hat{0}], [\hat{1}, \hat{1}] \notin [\![\mathcal{D}]\!]$ (see Example 50). Concerning $[\![\widehat{\mathcal{D}_{??\mathsf{d}}}]\!]$, notice that, since $k_0^0, k_0^1, k_0^2 \in \{0,1\}$, either $k_0^0 = k_0^1$ or $k_0^1 = k_0^2$ or $k_0^2 = k_0^0$. In the first case, we have $[[k_0^0], [k_0^1]] \in [\![\widehat{\mathcal{D}_{??\mathsf{d}}}]\!]$, in the second case we have $[[k_0^1], [k_0^2]] \in [\![\widehat{\mathcal{D}_{??\mathsf{d}}}]\!]$, and in the last case we have $[[k_0^2], [k_0^0]] \in [\![\widehat{\mathcal{D}_{??\mathsf{d}}}]\!]$. ◀

7 Conclusion and future work

For future research, we envisage extending our contributions in many directions. First, our notion of finite approximation seems intimately related with that of Taylor expansion from *differential linear logic* (DiLL) [15], where the rule hyp (quite like the rule 0 from DiLL) serves to model approximations of *boxes*. This connection with Taylor expansions becomes even more apparent in Mazza's original systems for parsimonious logic [25, 26], which comprise co-absorption and co-weakening rules typical of DiLL. These considerations deserve further investigations. Secondly, building on a series of recent works in *Cyclic Implicit Complexity*, i.e., implicit computational complexity in the setting of circular and non-wellfounded proof theory [10, 9], we are currently working on second-order extensions of wrPLL^∞ and rPLL^∞ to characterize the complexity classes **P**/poly and **P** (see [23]). These results would reformulate in a non-wellfounded setting the characterization of **P**/poly presented in [26].

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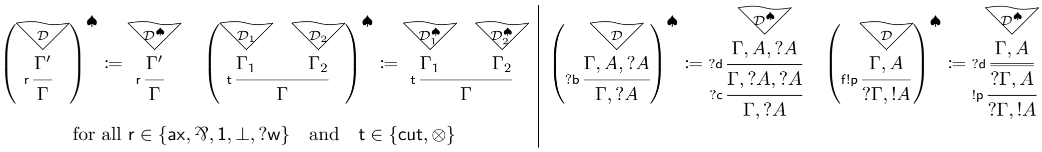


Figure 14 Translation $(\cdot)^\spadesuit$ from PLL to MELL.

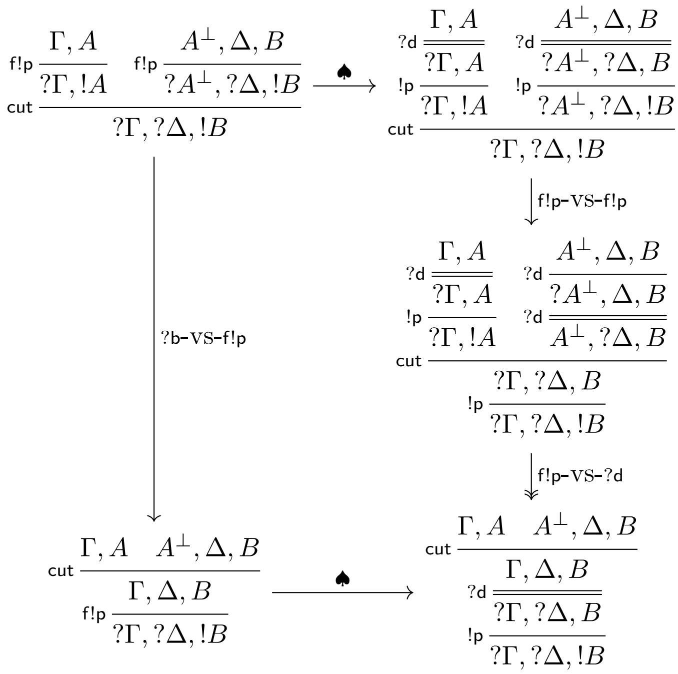


Figure 15 Commutation of the ?b-vs-f!p step and $(\cdot)^\spadesuit$.

A Appendix of Section 3

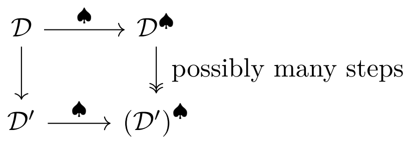
▶ **Theorem 5.** *For every* $\mathcal{D} \in \mathsf{PLL}$*, there is a cut-free* $\mathcal{D}' \in \mathsf{PLL}$ *such that* $\mathcal{D} \to^*_{\mathsf{cut}} \mathcal{D}'$.

Proof. We recall the sequent calculus for (propositional) *multiplicative exponential linear logic* $\mathsf{MELL} = \{\mathsf{ax}, \otimes, \parr, 1, \bot, \mathsf{cut}, !\mathsf{p}, ?\mathsf{w}, ?\mathsf{d}, ?\mathsf{c}\}$ where the **promotion** (!p), **dereliction** (?d), **contraction** (?c) rules are defined as follows:

$$\frac{?\Gamma, A}{?\Gamma, !A}\ !\mathsf{p} \qquad \frac{\Gamma, A}{\Gamma, ?A}\ ?\mathsf{d} \qquad \frac{\Gamma, ?A, ?A}{\Gamma, ?A}\ ?\mathsf{c} \tag{7}$$

We also denote by MELL the set of derivations over the rules in MELL, and we map each derivation in $\mathcal{D} \in \mathsf{PLL}$ to a derivation in $(\mathcal{D})^\spadesuit \in \mathsf{MELL}$ $(\cdot)^\spadesuit : \mathsf{PLL} \to \mathsf{MELL}$ defined in Figure 14 by induction on derivations.

In order to prove that the following diagram commute,



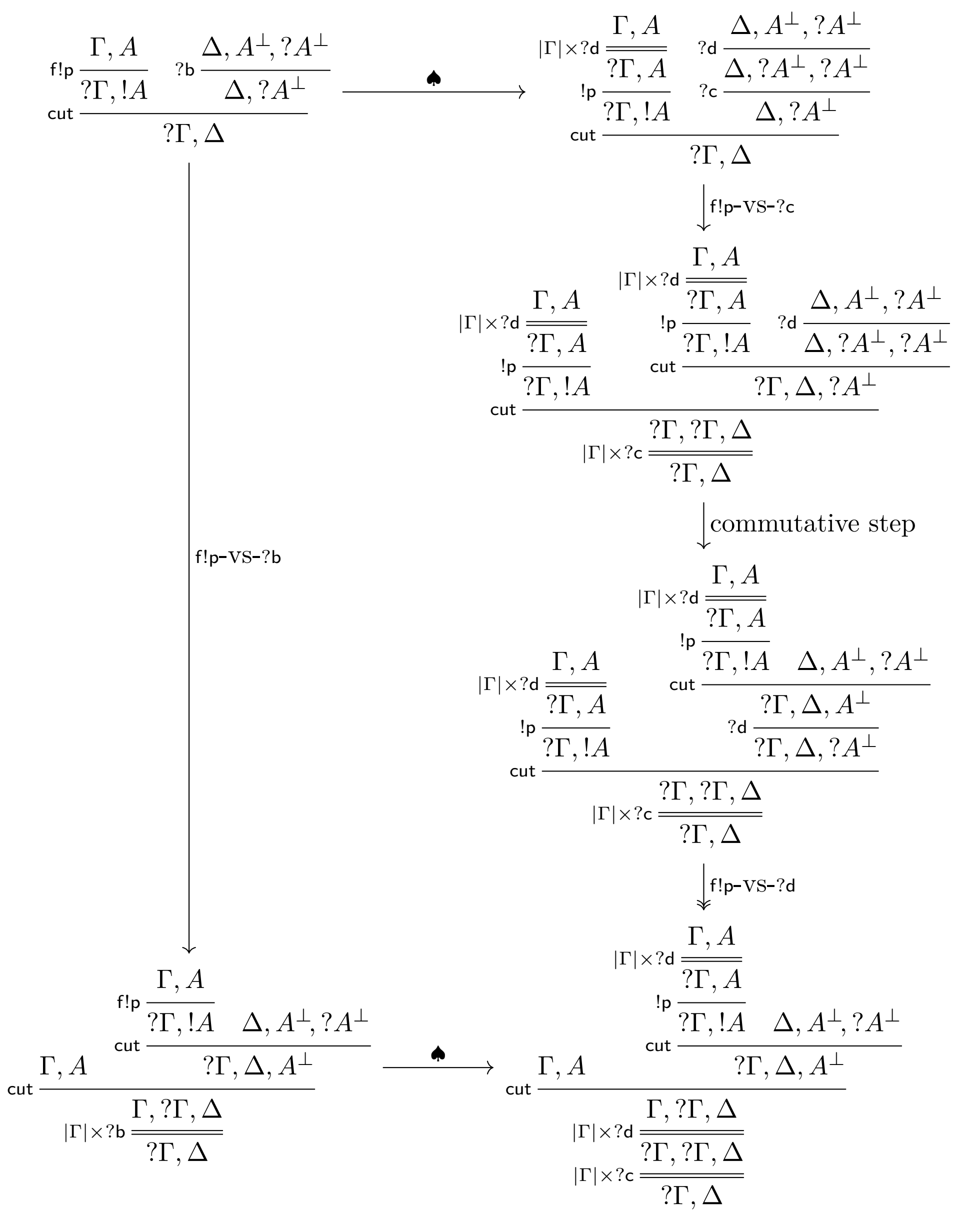


Figure 16 Commutation of the f!p-vs-?b step with $(\cdot)^{\spadesuit}$.

Each cut-elimination step in PLL corresponds to a cut-elimination step in MELL except the ones in Figures 15 and 16, where a cut-elimination step in PLL can be simulated by a sequence of cut-elimination steps in MELL. In these Figures each macro-step denoted by $\twoheadrightarrow$ involves a unique step from Figures 4 and 5 (the one marked) and certain additional commutative cut-elimination steps of the following form below

$$\mathsf{cut}\,\frac{\Gamma, A \qquad ?\mathsf{d}\,\dfrac{A^\perp, \Delta, B}{A^\perp, \Delta, ?B}}{\Gamma, \Delta, ?B} \to_{\mathsf{cut}} ?\mathsf{d}\,\frac{\mathsf{cut}\,\dfrac{\Gamma, A \qquad A^\perp, \Delta, B}{\Gamma, \Delta, B}}{\Gamma, \Delta, ?B} \qquad ?\mathsf{c}\,\frac{?\mathsf{d}\,\dfrac{\Gamma, A, ?B, ?B}{\Gamma, ?A, ?B, ?B}}{\Gamma, ?A, ?B} \to_{\mathsf{cut}} ?\mathsf{d}\,\frac{?\mathsf{c}\,\dfrac{\Gamma, A, ?B, ?B}{\Gamma, A, ?B}}{\Gamma, ?A, ?B} \tag{8}$$

which push ?d down a cut and create an alternating chain of ?d and ?c (such additional steps are natural to consider since they involve rule permutations of independent rules and would appear whenever a cut-rule would interact with the ?-formula introduced by the ?d-rule). Thus, the derivation in MELL obtained by (standard and additional) cut-elimination from $\mathcal{D}^{\spadesuit}$ is exactly the translation $(\mathcal{D}')^{\spadesuit}$ of the derivation $\mathcal{D}'$ in PLL obtained after a cut-elimination step from $\mathcal{D}$. According to the definition of $(\cdot)^{\spadesuit}$, if $(\mathcal{D}')^{\spadesuit}$ is cut-free then so is $\mathcal{D}'$.

The termination of cut-elimination in MELL with this additional commutative step follows from the result in MELL [29]. Indeed, to the usual measure m that decreases after each standard cut-elimination step in MELL (and remains unchanged after each additional step in (8)), we can add the sum d of the heights of the ?d rules in a derivation, which decreases after each step in (8). Thus, the measure (m, d) with the lexicographical order decreases after each (standard or additional) cut-elimination step in MELL. ◀