

Proof Diagrams for Multiplicative Linear Logic: Syntax and Semantics

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Abstract Proof nets are a syntax for linear logic proofs which gives a coarser notion of proof equivalence with respect to syntactic equality together with an intuitive geometrical representation of proofs. In this paper we give an alternative 2-dimensional syntax for multiplicative linear logic derivations. The syntax of string diagrams authorizes the definition of a framework where the sequentializability of a term, i.e. deciding whether the term corresponds to a correct derivation, can be verified in linear time. Furthermore, we can use this syntax to define a denotational semantics for multiplicative linear logic with units by means of equivalence classes of proof diagrams modulo a terminating rewriting.

Keywords String diagrams · Linear logic · Proof semantics · Monoidal categories

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1 Introduction

Proof nets are a geometrical representation of *linear logic* proofs introduced by Girard [8]. The building blocks of this syntax are called *proof structures*, later generalized by Lafont [19] in the so-called *interaction nets*. To recognize if a proof structure is a proof net one needs to verify its *sequentializability* property, that is, verifying whether it corresponds to a correct linear logic proof derivation.

Following Girard's original correction criterion, others methods have been introduced: the method by Danos-Regnier [6], that ensures graph acyclicity by a notion of *switchings* on \bowtie cells, and the method by Guerrini [12], that reformulates correction by means of graph contractibility. Unfortunately the aforementioned criteria become ineffective in presence of

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the multiplicative unit \perp . In order to recover a sequentialization condition for the multiplicative fragment with units (MLL_\perp) Girard has introduced the notion of *jumps* [11]. These are untyped edges which assign a \perp to an axiom in order to represent a dependency relation of the respective rules in sequentialization.

One peculiar feature of this syntax for proofs is that proof structures allow to recover the semantical equivalence of derivations under some inference rules permutations [19]. In the case of the multiplicative fragment of linear logic (MLL), proof nets perfectly capture this equivalence by giving a canonical representative for each class. On the other hand, in presence of multiplicative units, proof nets are not canonical [24] and have to be identified up to jump re-assignment, ruling out a satisfactory notion of proof net [16].

In this work we give an alternative syntax for MLL_\perp proofs and we study its properties. For this purpose, we replace the underlying interaction nets syntax with the one of string diagrams. We show that this syntax, which also presents an intuitive 2-dimensional representation of proofs, is able to capture some inference rule permutations in derivations.

String diagrams [18] are a syntax with a rigid structure for 2-arrows (or 2-cells) of a 2-category. Although the two syntaxes may graphically look similar, string diagrams' *strings* do not just denote connections between cells but they represent morphisms. Since crossing strings is not allowed without the introduction of *twisting operators*, we introduce the notion of *twisting relations* in order to equate diagrams by permitting cells to cross certain strings.

As soon as we consider a derivation of a proof as a sequence of n -ary operators applications over lists of formulas, we are able to express it by means of string diagrams which keep track of lists reordering. In a sense, string diagrams keep track of edge crossing in pictorial representations of proof nets.

We study several diagram rewriting systems given by *twisting polygraphs*. In this particular class of polygraph [5] string crossings are restrained to a specific family of strings, while some rewriting rules recover the graph representation equivalence.

The syntax of string diagrams allows us to define a polygraph where we introduce some *control strings* in order to encode the correct parenthesization of operators. In particular, these strings prevent the representation of non-correct applications of inference rules, resulting into a sound framework where sequentializability, that is whether a *proof diagram* corresponds to a derivation, can be checked in linear time on diagram inputs and outputs pattern only.

Furthermore, this syntax induces an equivalence relation over linear logic derivations representable by the same proof diagrams. However, this equivalence does not capture all rule permutations required for the elimination of the so called *commutative cuts*. In fact, these rule permutations require the permutation of derivation tree branches as shown in the following case:

$$\frac{\frac{\frac{1}{\vdots} \quad \frac{2}{\vdots} \quad \frac{3}{\vdots}}{\vdash \Gamma, A, B \quad \vdash \Delta, C} \otimes_1 \quad \vdash \Sigma, D}{\vdash \Gamma, \Delta, \Sigma, (A \otimes_1 D), (B \otimes_2 C)} \otimes_2 \quad \sim \quad \frac{\frac{1}{\vdots} \quad \frac{3}{\vdots} \quad \frac{2}{\vdots}}{\vdash \Gamma, A, B \quad \vdash \Sigma, D} \otimes_2 \quad \vdash \Delta, C}{\vdash \Gamma, \Delta, \Sigma, (A \otimes_1 D), (B \otimes_2 C)} \otimes_1$$

If a syntax does not equate derivations differing for rule permutations, it is crucial in view of *cut-elimination* theorem to explicitly authorize them. On the other hand, this syntax makes equivalent some proofs which are representable by proof nets differing in jumps assignment only.

With the purpose of keeping this last nice feature and extend the equivalence to include the missed rules permutations, we here extend the results presented in [3] by enriching our polygraph with some additional generators and rewriting rules. The equivalence induced by these rewriting rules induces an equivalence over derivation (seen as syntactical expressions)

effective to identify all and only MLL_U derivations which we use to consider equivalent (with respect of independent inference rules permutations).

In the last section we extend our polygraphic presentation with the rewriting rules for cut-elimination and a relative cut-elimination theorem. We conclude by giving a denotational semantics [9] for MLL_U proofs by means of equivalence classes of proof diagrams.

2 String Diagrams

In this section we recall some basic notions in string diagram rewriting [18]. For an introduction to this syntax see Selinger’s survey [23] and refer to John’s notes [4] for some interesting observations on the motivation and applications of this formalism.

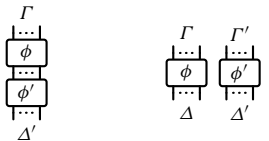
Given two lists $\Gamma = \Gamma_1 * \dots * \Gamma_n$ and $\Delta = \Delta_1 * \dots * \Delta_m$ of symbols in an alphabet Σ , a *string diagram* $\phi : \Gamma \Rightarrow \Delta$ with *inputs* $\text{in}(\phi) = \Gamma$ and *outputs* $\text{out}(\phi) = \Delta$ is pictured as follows:



A string diagram can be interpreted as a function with multiple inputs and outputs of type respectively $\Gamma_1, \dots, \Gamma_n$ and $\Delta_1, \dots, \Delta_m$. Diagrams may be composed in two different ways. If $\phi : \Gamma \Rightarrow \Delta$ and $\phi' : \Gamma' \Rightarrow \Delta'$ are diagrams, we define:

- *sequential* composition: if $\Delta = \Gamma'$, the diagram $\phi' \circ \phi : p \Rightarrow q'$ corresponds to usual composition of maps as the notation suggests. This composition is associative with units $\mathbf{id}_\Gamma : \Gamma \Rightarrow \Gamma$ for each possible list of inputs Γ . In other words, we have $\phi \circ \mathbf{id}_{\text{in}(\phi)} = \phi = \mathbf{id}_{\text{out}(\phi)} \circ \phi$. The *identity diagram* \mathbf{id}_Γ is pictured as follows: $\begin{matrix} \Gamma \\ | \dots | \\ \Gamma \end{matrix}$
- *parallel* composition: the diagram $\phi * \phi' : \Gamma * \Gamma' \Rightarrow \Delta * \Delta'$ is always defined. This composition is associative with unit $\mathbf{id}_\emptyset : \emptyset \Rightarrow \emptyset$. In other words, we have $\mathbf{id}_\emptyset * \phi = \phi = \phi * \mathbf{id}_\emptyset$. This \mathbf{id}_\emptyset is called the *empty diagram*.

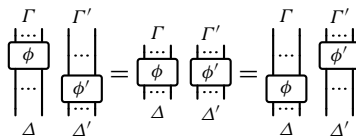
These two compositions are respectively represented as follows:



Our two compositions satisfy the *interchange rule*: if $\phi : \Gamma \Rightarrow \Delta$ and $\phi' : \Gamma' \Rightarrow \Delta'$, then

$$(\mathbf{id}_\Delta * \phi') \circ (\phi * \mathbf{id}_{\Gamma'}) = \phi * \phi' = (\phi * \mathbf{id}_{\Delta'}) \circ (\mathbf{id}_\Gamma * \phi')$$

that corresponds to the following picture:



String diagrams are a formalism for morphisms in a strict monoidal category with objects finite lists of symbols over an alphabet Σ . The sequential composition \circ denotes the usual morphisms composition while the product is the list concatenation and it is denoted by $*$.

Definition 1 (*Signature*) Fixed an alphabet Σ we denote by Σ^* the set of *words* or *lists* over Σ . A *signature* \mathcal{S} is a set of *atomic diagrams* (or *gates type*). Given a signature, a diagram $\phi : \Gamma \Rightarrow \Delta$ (with $\Gamma, \Delta \in \Sigma^*$) represents a morphism in the monoidal category \mathcal{S}^* in which morphisms are freely generated by \mathcal{S} , i.e. by the two compositions $*$ and \circ and identities. A *gate* is an occurrence of an atomic diagram, we denote $g : \alpha$ or we say that g is an α -gate if g is an occurrence of $\alpha \in \mathcal{S}$.

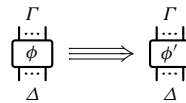
Definition 2 We say that ϕ is a *subdiagram* of ϕ' if and only if there exist $\chi_u, \chi_d \in \mathcal{S}^*$ and Γ, Δ such that $\phi' = \chi_d \circ (\mathbf{id}_\Gamma * \phi * \mathbf{id}_\Delta) \circ \chi_u$.

Notation Given $\phi \in \mathcal{S}^*$ and $\mathcal{S}' \subseteq \mathcal{S}$, we write $|\phi|_{\mathcal{S}'}$ the number of gates in ϕ with gate type $\alpha \in \mathcal{S}'$.

Definition 3 We call *horizontal* a diagram ϕ generated by parallel composition (and identities) only in \mathcal{S}^* . It is *elementary* if $|\phi|_{\mathcal{S}} = 1$.

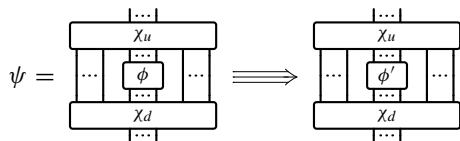
2.1 Diagram Rewriting

Definition 4 (*Diagram rewriting system*) Fixed an alphabet Σ , a *diagram rewriting system* is a couple $(\mathcal{S}, \mathcal{R})$ given by a signature \mathcal{S} and a set \mathcal{R} of rewriting rules of the form



where $\phi, \phi' : \Gamma \Rightarrow \Delta$ are diagrams in \mathcal{S}^* with same inputs and outputs. We call ϕ and ϕ' respectively *source* and *target* of the rewriting rules.

Definition 5 We allow each rewriting rule under any context, that is, if ϕ is a subdiagram of ψ and $\phi \Rightarrow \phi'$ in \mathcal{R} , then:



We say that ψ *reduces*, or *rewrites*, to ψ' (denoted $\psi \xRightarrow{*} \psi'$) if there is a *rewriting sequence* $P : \psi = \psi_0 \Rightarrow \psi_1 \Rightarrow \dots \Rightarrow \psi_n = \psi'$.

We here recall some classical notions in rewriting:

- A diagram ϕ is *irreducible* if there is no ϕ' such that $\phi \Rightarrow \phi'$;
- A rewriting system *terminates* if there is no infinite rewriting sequence;
- A rewriting system is *confluent* if for all ϕ_1, ϕ_2 and ϕ such that $\phi \Rightarrow \phi_1$ and $\phi \Rightarrow \phi_2$, there exists ϕ' such that $\phi_1 \xRightarrow{*} \phi'$ and $\phi_2 \xRightarrow{*} \phi'$;
- A rewriting system is *convergent* if both properties hold.

3 Polygraphs

In this section we formulate some basic notion in string diagram rewriting by using the language of *polygraphs*. Introduced by Street [25] as *computads*, later reformulated and extended by Burroni [5], polygraphs can be considered as the generalization for higher dimensional categories of the notion of monoid presentation and the construction of the free category generated by an oriented graph—i.e. the free category with objects the graph vertices and arrows the oriented paths [21].

In this paper we study some diagram rewriting systems with labels on strings in terms of 3-polygraphs, which are denoted $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)$ where each Σ_i is the set of its i -cells. In particular, we consider polygraphs with just one 0-cell in Σ_0 ; otherwise we would need to introduce background labelings in string diagrams. The set of 1-cells Σ_1 represents string labels, the 2-cells in Σ_2 are the signature S_Σ of our rewriting system with rules $\mathcal{R}_\Sigma = \Sigma_3$, the set of 3-cells. We say that a polygraph Σ exhibits some computational properties when the relative diagram rewriting system does.

Notation We denote $\phi \in \Sigma$ whenever ϕ is a diagram generated by the associated signature S_Σ . If Σ is a 3-polygraph with one 0-cell, we denote by $\langle \Sigma \rangle$ the *monoidal category generated by Σ* , that is the category with objects the words in Σ_1 and morphisms $[\phi]$ the equivalence classes of diagrams $\phi \in S^*$ modulo \mathcal{R}_Σ (we denote $[\phi] \in \langle \Sigma \rangle$). For a detailed description of the construction of the category generated by a polygraph refer to [5, 15]. We say that Σ' extends Σ if Σ' can be obtained by Σ by extending the sets of i -cells, that is $\Sigma_i \subseteq \Sigma'_i$ for all i .

3.1 Twisting Polygraph

In this section we introduce a notion of polygraph which generalizes polygraphic presentations of symmetric monoidal categories.

Definition 6 (*Symmetric polygraph*) We call the *polygraph of permutation* the following monochrome 3-polygraph:

$$\mathfrak{S} = \left(\Sigma_0 = \{\square\}, \Sigma_1 = \{\parallel\}, \Sigma_2 = \{\bowtie\}, \Sigma_3 = \left\{ \begin{array}{c} \text{X} \Rightarrow \text{X} \\ \text{X} \Rightarrow \text{X} \end{array} \right\} \right)$$

We call *symmetric* a 3-polygraph Σ with one 0-cell, one 1-cell (i.e. $\Sigma_1 = \{\parallel\}$), containing one 2-cell $\bowtie \in \Sigma_2$ and such that the following holds

$$\begin{array}{c} \text{X} \\ \text{X} \end{array} = \parallel, \quad \begin{array}{c} \alpha \\ \text{X} \\ \text{X} \\ \alpha \end{array} = \begin{array}{c} \text{X} \\ \alpha \\ \text{X} \\ \text{X} \\ \alpha \end{array} \quad \text{and} \quad \begin{array}{c} \text{X} \\ \alpha \\ \text{X} \\ \text{X} \\ \alpha \end{array} = \begin{array}{c} \text{X} \\ \text{X} \\ \alpha \\ \text{X} \\ \alpha \end{array} \quad \text{for all } \alpha \in \Sigma_2$$

in the 2-category Σ^* . In such 3-polygraph to denote diagrams inputs and outputs it suffices to provide the respective numbers of their input and output strings.

Theorem 1 (Convergence of \mathfrak{S}) *The polygraph \mathfrak{S} is convergent.*

Proof As in [20], in order to prove termination we interpret every diagram $\phi : n \Rightarrow m \in \mathfrak{S}^*$ with a monotone function $[\phi] : \mathbb{N}^n \rightarrow \mathbb{N}^m$. These have well-founded partial order induced by product order on \mathbb{N}^p ($\bar{x} = (x_1, \dots, x_p) \leq (y_1, \dots, y_p) = \bar{y}$ whenever $x_1 \leq x_1 \wedge \dots \wedge x_p \leq y_p$):

$$f, g : \mathbb{N}^{*p} \rightarrow \mathbb{N}^{*p} \text{ then } f < g \text{ iff } f(\bar{x}) < g(\bar{x}) \text{ for all } \bar{x} \in \mathbb{N}^{*p}.$$

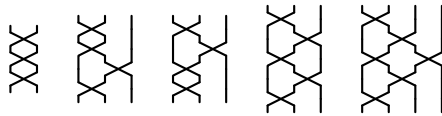
We interpret the gate \bowtie by the function $[\bowtie](x, y) \rightarrow (x + y, x)$. This allows us to associate to any 3-cell $\phi \Rightarrow \psi$ two monotone maps $[\phi]$ and $[\psi]$ such that $[\phi] > [\psi]$:

$$[\begin{array}{c} \bowtie \\ \hline \end{array}](x, y) = (2x + y, x + y) > (x, y) = [\begin{array}{c} | \quad | \\ \hline \end{array}](x, y),$$

$$[\begin{array}{c} \bowtie \\ \hline \bowtie \\ \hline \end{array}](x, y, z) = (2x + y + z, x + y, x) > (x + y + z, x + y, x) = [\begin{array}{c} \bowtie \\ \hline \bowtie \\ \hline \end{array}](x, y, z)$$

By the compatibility of the order with sequential and parallel composition, this suffices to prove that, for any couple of diagrams, $[\phi] > [\psi]$ holds if $\phi \xrightarrow{*} \psi$. Since this order on monotone maps on integers admits no infinite decreasing chain, infinite reduction paths can not exist.

In order to prove convergence, it suffices to check the confluence of the following critical peaks, that are the minimal critical branchings of the rewriting system (see [2] for details):



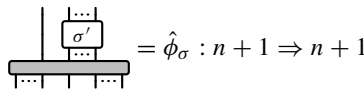
□

Each diagram in \mathfrak{S} can be interpreted as a permutation in the group of permutations over n elements S_n with product \circ defined as their function composition. On the other hand, each $\sigma \in S_n$ corresponds to some diagrams in \mathfrak{S} . In particular, we interpret the diagram $\mathbf{id}_{k-1} * \bowtie * \mathbf{id}_{n-(k+1)} : n \Rightarrow n$ as the transposition $(k, k + 1) \in S_n$.

Notation We note $Lad_n^l = \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \bowtie \begin{array}{c} \dots \\ \dots \\ \dots \end{array} : n \Rightarrow n$ and $Lad_n^r = \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \bowtie : n \Rightarrow n$ the left and right ladder diagrams corresponding respectively to the permutations $(2, 3, \dots, n, 1)$ and $(n, 1, 2, \dots, n - 1)$ in S_n .

Proposition 1 *For any permutation $\sigma \in S_n$ there is a unique diagram in normal form $\hat{\phi}_\sigma : n \Rightarrow n \in \mathfrak{S}$ corresponding to σ . We call it the canonical diagram of σ .*

Proof We define $\mathfrak{S}_1 = \{|\}$ and \mathfrak{S}_{n+1} the set of diagrams in \mathfrak{S} of the form:



with $\begin{array}{c} \dots \\ \dots \\ \dots \end{array} \sigma' \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \in \mathfrak{S}_n$ and $\begin{array}{c} \dots \\ \dots \\ \dots \end{array} \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \begin{array}{c} \dots \\ \dots \\ \dots \end{array} = \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \bowtie \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \begin{array}{c} \dots \\ \dots \\ \dots \end{array} = Lad_k^l * id_{(n+1-k)}$. We have $|\mathfrak{S}_n| = n!$ since $|\mathfrak{S}_1| = 1$ and $|\mathfrak{S}_{n+1}| = (n + 1)|\mathfrak{S}_n|$ on account of $n + 1 = |\{Lad_k^l\}_{1 \leq k \leq n+1}| = |\{Lad_k^l * \mathbf{id}_{(n+1-k)}\}_{1 \leq k \leq n+1}|$.

To exhibit a one-to-one correspondence between S_{n+1} and \mathfrak{S}_{n+1} , for any $\sigma \in S_{n+1}$ we define $Er(\sigma) \in S_n$ as the permutation

$$Er(\sigma) = \begin{cases} i \rightarrow \sigma(i + 1) & \text{if } \sigma(i + 1) < \sigma(1) \\ i \rightarrow \sigma(i + 1) + 1 & \text{if } \sigma(1) < \sigma(i + 1) \end{cases}$$

and $\hat{\phi}_\sigma = (Lad_k^l * \mathbf{id}_{(n+1-\sigma(1))}) \circ (\mathbf{id}_1 * \hat{\phi}_{Er(\sigma)})$.

No element in \mathfrak{S}_n contains subdiagram of the form nor This means that they are irreducible and so, by the confluence of \mathfrak{S} , in normal form. □

Definition 7 (*Twisting polygraph*) A *twisting polygraph* is a 3-polygraph Σ with one 0-cell equipped with a set $T_\Sigma \subseteq \Sigma_1$ called *twisting family* such that for each $A, B \in T_\Sigma$ there is a *twisting operator* $\bowtie_{A,B} : A * B \Rightarrow B * A \in \Sigma_2$ and Σ_3 includes the following families $T_{\mathcal{R}}$ of *twisting relations*:

- For all $A, B, C \in T_\Sigma$:

$$\begin{array}{c} A & B \\ \diagdown & / \\ & \times \\ / & \diagdown \\ A & B \end{array} \iff \begin{array}{c} | & | \\ A & B \\ | & | \end{array} \quad \text{and} \quad \begin{array}{c} A & B & C \\ \diagdown & / & \\ & \times & \\ / & \diagdown & \\ C & B & A \end{array} \iff \begin{array}{c} A & B & C \\ \diagdown & / & \\ & \times & \\ / & \diagdown & \\ C & B & A \end{array}; \quad (1)$$

- For all $\alpha : \Gamma \rightarrow \Gamma' \in \Sigma_2$ with $\Gamma, \Gamma' \in T_\Sigma^*$, $A \in T_\Sigma$, at least one of the two possible orientation of the following rewriting rules is in Σ_3 :

$$\begin{array}{c} \Gamma & A \\ \square \alpha \\ \Gamma' \end{array} \iff \begin{array}{c} \Gamma & A \\ \square \alpha \\ \Gamma' \end{array} \quad \text{and} \quad \begin{array}{c} A & \Gamma \\ \square \alpha \\ \Gamma' & A \end{array} \iff \begin{array}{c} A & \Gamma \\ \square \alpha \\ \Gamma' & A \end{array}. \quad (2)$$

Moreover, if ϕ, ψ are *twisting diagrams* (i.e. diagrams made only of twisting operators) $\phi \stackrel{*}{\sim}_{\mathcal{R}_\Sigma} \psi$ iff $\phi \stackrel{*}{\sim}_{\mathcal{R}_T} \psi$ where \mathcal{R}_T is the set given by rewriting rules of (1). A *total-twisting polygraph* is a twisting polygraph with $T_\Sigma = \Sigma_1$.

The idea behind twisting polygraphs is to present diagram rewriting systems where, in equivalence classes modulo rewriting, the crossings of strings labeled by the twisting family are not taken into account. In fact, the family of relations (1) says that these crossings are involutive and satisfy Yang–Baxter equation [17] for braidings, while relations in (2) allow gates to “cross” a string in case of fitting labels.

We interpret a twisting diagram $\phi_\sigma : \Gamma \Rightarrow \sigma(\Gamma)$ as the permutations in $S_{|\Gamma|}$ acting over the order of occurrence of 1-cells in the word $\Gamma \in T_\Sigma^*$. For this reason, as in \mathfrak{S} , we define left ladders, right ladders and the standard diagrams $\hat{\phi}_\sigma^\Gamma : \Gamma \rightarrow \sigma(\Gamma)$ (or simply $\hat{\phi}_\sigma$) with source and target in T_Σ^* . In conformity with the twisting polygraph restrictions over Σ_3 , we can prove the uniqueness of $\hat{\phi}_\sigma$ as in Proposition 1.

4 Multiplicative Linear Logic Sequent Calculus

In this paper we focus on the multiplicative fragment of linear logic sequent calculus with units. We here we recall the usual inference rules:

- Structural rules are the *axiom* and the *cut*:

$$\frac{}{\vdash A, A^\perp} Ax \quad \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} Cut$$

- Multiplicative rules are the *tensor* (\otimes) and the *par* (\wp):

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \otimes \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \wp$$

– (Multiplicative) units are the *bottom* (\perp) and the *one* (1):

$$\frac{\vdash \Gamma, A, \perp}{\vdash \Gamma, A} \perp \qquad \frac{}{\vdash 1} 1$$

We also consider the usually omitted exchange rule:

$$\frac{\vdash A_1, \dots, A_k}{\vdash A_{\sigma(1)}, \dots, A_{\sigma(k)}} \sigma \in S_k$$

We call *principal* a formula which occurs in the conclusion of a rule but does not occur in the premise(s) and *active* a formula which occurs in the premise of a rule but not in the conclusion. In a derivation $d(\Gamma)$, we say that a *Cut* rule is *commutative* when one of its active formulas is not principal. Moreover, a commutative cut is *pure* if the non-principal active formula is principal for a \otimes rule. A *cut-free derivation* is a derivation with no occurrences of *Cut* rules.

We finally recall that the *multiplicative linear logic fragment with units* (MLL_U) is given by the aforementioned inference rules, while the *multiplicative fragment* (MLL) is the one given by the inference rules Ax , *Cut*, \otimes , \wp (and exchange) only.

Definition 8 (*Negation*) We assume negation to be involutive, i.e. $A^{\perp\perp} = A$ and the De Morgan’s laws to apply with respect to \wp and \otimes , i.e. $(A \wp B)^{\perp} = B^{\perp} \otimes A^{\perp}$ and $(A \otimes B)^{\perp} = A^{\perp} \wp B^{\perp}$ for any formulas A, B . Moreover $1^{\perp} = \perp$.

Remark 41 (*On rules*) In this work we interpret inference rules as operators with specific arities over the set of sequents: Ax and 1 are nullary, \wp and \perp are unary and \otimes and *Cut* are binary.

Notation We denote with $\mathfrak{F}_{\text{MLL}}$ and $\mathfrak{F}_{\text{MLL}_U}$ the sets of formulas respectively in MLL and MLL_U . Moreover we denote with $\mathfrak{F}_{\text{MLL}}^*$ and $\mathfrak{F}_{\text{MLL}_U}^*$ their respective sets of sequents.

In the formalism of sequent calculus, oftentimes two derivations are identified when they can be transformed one into the other by a sequence of permutations over inference rules. Indeed, this identification is crucial to obtain a cut-elimination result whenever we face a commutative cut. In this paper we consider the equivalence among derivations only from a syntactical viewpoint: namely, two derivations are considered equal if and only if they display exactly the same sequents (multisets of formulas) and the same rules in the same order. We then formalize the equivalence relation \sim over MLL_U derivations given by the permutation of inference rules with disjoint sets of active formula occurrences:

Definition 9 We define the *standard equivalence over MLL_U derivations* (denoted by \sim) as the equivalence relation generated by the following equivalences for all $A, B, C, D \in \mathfrak{F}_{\text{MLL}_U}$, $\Gamma, \Delta, \Sigma \in \mathfrak{F}_{\text{MLL}_U}^*$ and for all permutations of the branching order of derivations:

$$\frac{\frac{\vdash A, B, C, D, \Gamma}{\vdash A \wp B, C, D, \Gamma} \wp}{\vdash A \wp B, C \wp D, \Gamma} \wp \sim \frac{\frac{\vdash A, B, C, D, \Gamma}{\vdash A, B, C \wp D, \Gamma} \wp}{\vdash A \wp B, C \wp D, \Gamma} \wp \qquad \frac{\frac{\vdash A, B, \Gamma}{\vdash A, B, \perp, \Gamma} \perp}{\vdash A \wp B, \perp, \Gamma} \wp \sim \frac{\frac{\vdash A, B, \Gamma}{\vdash A \wp B, \Gamma} \wp}{\vdash A \wp B, \perp, \Gamma} \perp$$

$$\frac{\frac{\vdash \Gamma}{\perp_1, \Gamma} \perp_1}{\perp_1, \perp_2, \Gamma} \perp_2 \sim \frac{\frac{\vdash \Gamma}{\perp_2, \Gamma} \perp_2}{\perp_1, \perp_2, \Gamma} \perp_1 \qquad \frac{\frac{\vdash A, C, D, \Gamma \vdash B, \Delta}{\vdash A \otimes B, C, D, \Gamma, \Delta} \otimes}{\vdash A \otimes B, C \wp D, \Gamma, \Delta} \wp \sim \frac{\frac{\vdash A, C, D, \Gamma}{\vdash A, C \wp D, \Gamma} \wp}{\vdash A \otimes B, C \wp D, \Gamma} \wp \vdash B, \Delta$$

$$\frac{\frac{\frac{\vdash A, \Gamma \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \otimes}{\vdash A \otimes B, \perp, \Gamma, \Delta} \perp}{\vdash A \otimes B, \perp, \Gamma, \Delta} \perp \sim \frac{\frac{\vdash A, \Gamma}{\vdash A, \perp, \Gamma} \perp}{\vdash A \otimes B, \perp, \Gamma} \otimes \vdash B, \Delta$$

$$\frac{\frac{\frac{\vdash A, C, \Gamma \vdash D, \Delta}{\vdash A, C \otimes D, \Gamma, \Delta} \otimes}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta} \otimes}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta} \otimes \vdash B, \Sigma \sim \frac{\frac{\frac{\vdash A, C, \Gamma \vdash B, \Sigma}{\vdash A \otimes B, C, \Gamma, \Sigma} \otimes}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma} \otimes}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma} \otimes \vdash D, \Delta$$

together with the not written equivalences for *Cut*, which are analogous to the \otimes relations (with suitable premises and conclusions).

We define the *cut-elimination procedure* by the following set of rewriting rules over derivations:

Definition 10 (*Cut-elimination procedure*) The cut-elimination procedure is the relation \rightarrow_{Cut} generated by the following (oriented) relations called *cut-elimination steps*:

$$\frac{\frac{\vdash \overset{\cdot}{\Gamma}, A \quad \frac{\vdash A, A^\perp}{\vdash \Gamma, A} Ax}{\vdash \Gamma, A} Cut}{\vdash \Gamma, A} \rightarrow_{Cut} \vdash \overset{\cdot}{\Gamma}, A \quad \frac{\frac{\vdash A, A^\perp}{\vdash \Gamma, A} Ax \quad \vdash \overset{\cdot}{\Gamma}, A}{\vdash \Gamma, A} Cut \rightarrow_{Cut} \vdash \overset{\cdot}{\Gamma}, A$$

$$\frac{\frac{\frac{\vdash \overset{\cdot}{\Gamma}, A \quad \vdash \overset{\cdot}{\Delta}, B}{\vdash A \otimes B, \Gamma, \Delta} \otimes \quad \frac{\vdash B^\perp, A^\perp, \Sigma}{\vdash B^\perp \wp A^\perp, \Sigma} \wp}{\vdash \Gamma, \Delta, \Sigma} \otimes \wp}{\vdash \Gamma, \Delta, \Sigma} \rightarrow_{Cut} \frac{\vdash \overset{\cdot}{\Gamma}, A \quad \frac{\vdash \overset{\cdot}{\Delta}, B \quad \vdash B^\perp, A^\perp, \Sigma}{\vdash \Delta, A, \perp, \Sigma} Cut}{\vdash \Gamma, \Delta, \Sigma} Cut$$

$$\frac{\frac{\frac{\vdash B^\perp, A^\perp, \Sigma}{\vdash B^\perp \wp A^\perp, \Sigma} \wp \quad \frac{\vdash \overset{\cdot}{\Gamma}, A \quad \vdash \overset{\cdot}{\Delta}, B}{\vdash A \otimes B, \Gamma, \Delta} \otimes}{\vdash \Gamma, \Delta, \Sigma} \wp \otimes}{\vdash \Gamma, \Delta, \Sigma} \rightarrow_{Cut} \frac{\frac{\vdash B^\perp, A^\perp, \Sigma \quad \vdash \overset{\cdot}{\Gamma}, A}{\vdash B^\perp, \Gamma, \Sigma} Cut \quad \vdash \overset{\cdot}{\Delta}, B}{\vdash \Gamma, \Delta, \Sigma} Cut$$

$$\frac{\frac{\vdash \overset{\cdot}{\Gamma}}{\vdash \Gamma, \perp} \perp \quad \frac{\vdash 1}{\vdash 1} 1}{\vdash \Gamma} \rightarrow_{Cut} \vdash \overset{\cdot}{\Gamma} \quad \frac{\frac{\vdash 1}{\vdash 1} 1 \quad \frac{\vdash \overset{\cdot}{\Gamma}}{\vdash \Gamma, \perp} \perp}{\vdash \Gamma} \rightarrow_{Cut} \vdash \overset{\cdot}{\Gamma}, A$$

The cut-elimination theorem for MLL_U sequent calculus is proved by showing the termination of the cut-elimination procedure [8]. This result requires the identification of derivations by the standard equivalence. Alternatively, the proof requires the definition of some additional rewriting rules which permute the commutative *Cut* instances. We remark that even in non-commutative extensions of linear logics [1] where permutations of formulas in a sequent are strongly restricted, we require permutations over derivation branches for a proof of cut-elimination theorem.

For this reason, any *denotational semantic* of MLL_U sequent calculus [7, 10, 22] has to take into account the standard equivalence of derivations in order to capture the cut-elimination. It results that the equivalence relation over derivations induced by any such semantics contains the equivalence relation \approx over the derivations syntax generated by $(\rightarrow_{Cut} \cup \sim)$.

5 String Diagram Syntax for Linear Logic

In this section we define some particular 3-polygraphs which generate a family of string diagrams we call *proof diagrams*. These diagrams are a syntax for linear logic sequent calculus with explicit exchange rules.

The first polygraph MPN_{MLL_U} we define generates a family of terms corresponding to the different representations of MLL_U proof nets, with explicit notation for wire crossings but no jump assignments.

We then improve this construction adding two non-twisting colors for strings and we adapt certain gate types in order to make them interact with these *control strings*. Due to the more rigid structure of the diagrammatic syntax, in this polygraph $\tilde{\mathcal{U}}$ we are able to characterize diagrams corresponding to linear logic derivations by just checking their inputs and outputs patterns. On the other hand, the rewriting we define is able to capture all permutations of inference rules with exception of the ones between two binary rules (\otimes or *Cut*), in particular the one needed to eliminate commutative cuts, crucial for the sequent calculus cut-elimination theorem.

We extend to \mathcal{U} the polygraphic presentation of this model by extending $\tilde{\mathcal{U}}$ with two sets of generators and relations which allows us to perform some transformations corresponding to certain permutations of binary inference rules. We then show that the classes of equivalent diagrams modulo the rewriting of this polygraph are in one-to-one correspondence with the classes of \sim -equivalent $\text{MLL}_{\mathcal{U}}$ proof.

We conclude with the polygraph \mathcal{U}_{Cut} which include the rewriting rules corresponding to cut-elimination steps of $\text{MLL}_{\mathcal{U}}$ sequent calculus showing that the associated quotient over $\text{MLL}_{\mathcal{U}}$ derivations captures the semantic equivalence of proof.





Notation From now on, in order to unify the notation 1-cell composition with the one of sequents, we replace the symbol $*$ for string diagrams parallel composition with a comma.

5.1 Proof Diagrams for $\text{MLL}_{\mathcal{U}}$ Proof Nets

The first polygraph we introduce can be seen as a formal syntax for proof net representations.

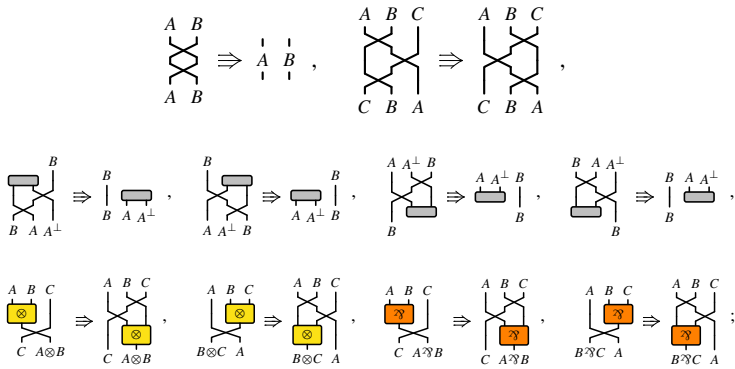
Definition 11 The 3-polygraph MPN is the *polygraph of multiplicative linear logic proof nets with units*. It is given by the following sets of cells:

$$\begin{aligned}
 & - \text{MPN}_0 = \{ \square \}; \\
 & - \text{MPN}_1 = \left\{ \begin{array}{c} | \\ A \end{array} \right\}_{A \in \tilde{\mathfrak{F}}_{\text{MLL}_{\mathcal{U}}}}; \\
 & - \text{MPN}_2 = \left(\begin{array}{l} \otimes_{A,B} : A, B \Rightarrow A \otimes B = \begin{array}{c} A \ B \\ \otimes \\ A \otimes B \end{array} \\ \wp_{A,B} : A, B \Rightarrow A \wp B = \begin{array}{c} A \ B \\ \wp \\ A \wp B \end{array} \\ Ax_A : \square \Rightarrow A, A^\perp = \begin{array}{c} A \\ A \ A^\perp \end{array} \\ Cut_A : A, A^\perp \Rightarrow \square = \begin{array}{c} A \ A^\perp \\ A \end{array} \\ \times_{A,B} : A, B \Rightarrow B, A = \begin{array}{c} A \ B \\ \times \\ B \ A \end{array} \\ 1 : \square \Rightarrow 1 = \begin{array}{c} \circ \\ 1 \end{array} \\ \perp : \square \Rightarrow \perp = \begin{array}{c} \bullet \\ \perp \end{array} \end{array} \right)_{A, B \in \tilde{\mathfrak{F}}_{\text{MLL}_{\mathcal{U}}}}
 \end{aligned}$$

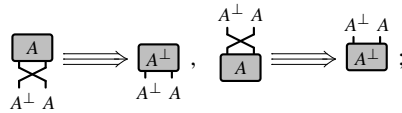
If there is no ambiguity we note  and  instead of  and .

$$- \text{MPN}_3 = \text{MPN}_{\text{Twist}}^M \cup \text{MPN}_{\text{Twist}}^u \cup \text{MPN}_{\text{Cut}}^{Ax} \cup \text{MPN}_{\text{Cut}}^M \cup \text{MPN}_{\text{Cut}}^u \text{ where }$$

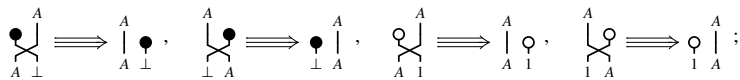
– MPN_{Twist}^M is given by the following twisting relations:



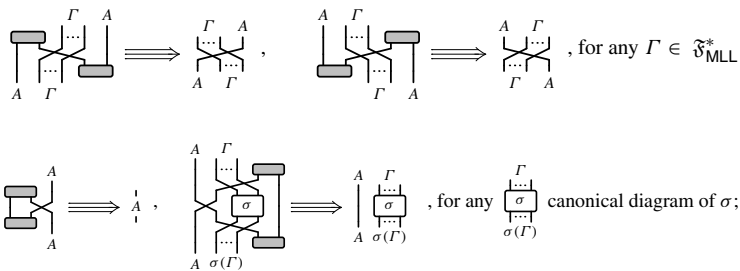
together with two rules representing the involution $A^{\perp\perp} = A$:



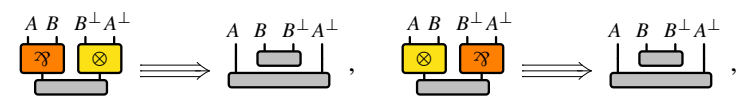
– MPN_{Twist}^u is given by the following twisting relations:



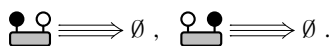
– MPN_{Cut}^{Ax} is following the set of rules for the cut elimination:



– MPN_{Cut}^M is following the set of rules for the cut elimination:



– MPN_{Cut}^u the following set of rules for cut elimination:



Remark 51 The polygraph **MPN** is twisting with twisting family $\mathfrak{F}_{\text{MLL}_u}$, i.e. it is total twisting.

Theorem 2 (Interpretation of proofs in **MPN**) *For any derivation $d(\Gamma)$ of $\vdash \Gamma$ in MLL_u there is a proof diagram $\phi_{d(\Gamma)} : \square \Rightarrow \Gamma \in \text{MPN}$.*

Proof Let $d(\Gamma)$ be a derivation in MLL_u of $\vdash \Gamma$. First we observe that, if there is a diagram $\phi : \Delta \Rightarrow \Gamma$ so there also is a diagram $\phi^\sigma = \hat{\phi}_\sigma \circ \phi : \Delta \Rightarrow \sigma(\Gamma)$ for all permutation $\sigma \in S_{|\Gamma|}$. Thus, we can proceed by induction on the number of inference rules appearing in $d(\Gamma)$:

- If just one inference rule occurs in $d(\Gamma)$, it must be an Ax rule or a 1 rule. It follows that $\Gamma = A, A^\perp$ and $\phi_{d(\Gamma)} = Ax_A : \square \Rightarrow A, A^\perp$ or that $\Gamma = 1$ and $\phi_\Gamma = 1 : \square \Rightarrow 1$;
- If $n + 1$ inference rules occur in $d(\Gamma)$, then we consider the last one and we distinguish two cases in base of its arity (see Remark 41):
- If it is unary and $\Gamma = \Gamma', A \wp B$, then, by inductive hypothesis, there is a diagram $\phi_{d(\Gamma', A, B)} : \square \Rightarrow \Gamma', A, B$ of the derivation $d(\Gamma', A, B)$ with n inference rules. Therefore

$$\phi_{d(\Gamma)} = (\mathbf{id}_{\Gamma'}, \wp_{A, B}) \circ \phi_{d(\Gamma', A, B)} : \square \Rightarrow \Gamma;$$

- If it is an unary \perp and $\Gamma = \Gamma', \perp$, then, by inductive hypothesis, there is a diagram $\phi_{\Gamma'} : \square \Rightarrow \Gamma'$ and $\phi_\Gamma = \phi_{\Gamma'}, \perp$.
- If it is binary and $\Gamma = \Delta, A \otimes B, \Delta'$, then, by inductive hypothesis, there are two diagrams $\phi_{d(\Delta, A)} : \square \Rightarrow \Delta, A$ and $\phi_{d(B, \Delta')} : \square \Rightarrow B, \Delta'$ relative to the two derivations $d(\Delta, A)$ and $d(B, \Delta')$ with at most n inference rules. Therefore

$$\phi_{d(\Gamma)} = (\mathbf{id}_\Delta, \otimes_{A, B}, \mathbf{id}_{\Delta'}) \circ (\phi_{d(\Delta, A)}, \phi_{d(B, \Delta')}) : \square \Rightarrow \Gamma;$$

- Similarly, if it is binary and $\Gamma = \Delta, Cut(A, A^\perp), \Delta'$, then

$$\phi_{d(\Gamma)} = (\mathbf{id}_\Delta, cut_A, \mathbf{id}_{\Delta'}) \circ (\phi_{d(\Delta, A)}, \phi_{d(A^\perp, \Delta')}) : \square \Rightarrow \Gamma.$$


□

The 2-cells of this syntax reminds MLL_u proof structure representations. We remark two important differences: cells are always top-to-bottom orientated, that is with the active port on the bottom, and wire crossing are part of this syntax by means of twisting operators. This intuition leads to the following:

Proposition 2 (Proof structure interpretation) *We can associate to any proof diagram ϕ in **MPN** a MLL_u proof structures P_ϕ .*

Proof It suffices to consider a proof diagram as a specific representation of a proof structure with no specific jumps assignment: strings, Ax -gates and Cut -gates are interpreted as wires ($\cap \rightsquigarrow \text{gate}$ and $\cup \rightsquigarrow \text{gate}$), twisting operators as wire crossing and gates of type \otimes, \wp, \perp and 1 as the corresponding cells of the proof structure with a coherent labeling with respect of gate types. Then, since proof diagrams in **MPN** keep no records about jump assignments, for each \perp -gate we assign arbitrary jump. □

However, the converse is not true. In fact even if we interpret down-to-down and up-to-up wire turn-backs as Ax and Cut gates respectively (i.e. $\cap \rightsquigarrow \text{gate}$ and $\cup \rightsquigarrow \text{gate}$) and wire crossing as occurrences of twisting operators, in the syntax of proof diagrams we are not able

to represent some (incorrect) proof structures because of the type of inputs and outputs of Ax and Cut gates. By means of example, consider the proof structure  whose translation in proof diagram syntax requires the existence of $A, B \in \mathfrak{F}_{\text{MLL}_U}$ with $A^\perp = A^\perp \wp B$ in order to be well defined.

For this syntax we do not formulate any correctness criterion since this would be a reformulations of a standard proof net one.

5.2 Proof Diagram with Control for MLL_U

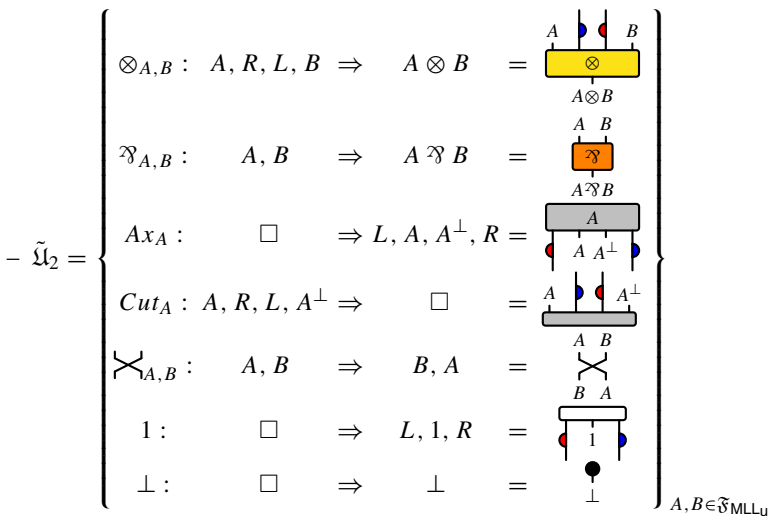
In order to have an analogous of the proof net correctness criterion formalized inside a syntax of MLL_U proof diagrams, we enrich the set of string labels with two new non-twisting colors $L = \blacktriangleleft$ (left) and $R = \blacktriangleright$ (right) that we call *control strings*.

The idea is to use these strings to reproduce a 2-dimensional notation for parenthesization, in order to internalize a notion of well-parenthesization in a setting where a proof derivation can be seen as a sequence of operations over lists of sequents. Thus, unary derivation rules act on single sequents (as in the case of \wp and \perp), binary ones act on two sequent (as in the case of \otimes and Cut) and the nullary one, that are Ax and 1 , generates a new sequent. For this purpose we re-define the 2-cells for 0-ary and binary rules in order to make them interact with control strings.

Notation In order to help reader, L and R control strings are represented in diagrams by strings decorated by a certain number of \blacktriangleleft and \blacktriangleright respectively. These have to be considered as string labels and not gates.

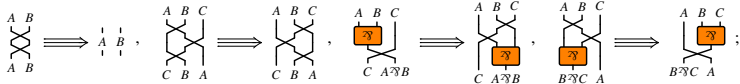
Definition 12 The *control polygraph of multiplicative linear logic with units* $\tilde{\mathfrak{M}}$ is given by the following sets of cells:

- $\tilde{\mathfrak{M}}_0 = \{ \square \};$
- $\tilde{\mathfrak{M}}_1 = \{ \text{ } \}_{A \in \mathfrak{F}_{\text{MLL}_U}} \cup \{L = \blacktriangleleft, R = \blacktriangleright\};$

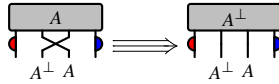


- $\tilde{\mathfrak{M}}_3 = \tilde{\mathfrak{M}}_{Twist} \cup \tilde{\mathfrak{M}}_{Twist}$ where:

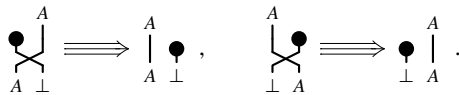
- $\tilde{\mathfrak{M}}_{Twist}$ is given by the following twisting relations:



together with one rule representing the involution $A^{\perp\perp} = A$:



– $\tilde{\mathfrak{U}}_{Twist}$ is given by the following twisting relations:



Remark 52 The polygraph $\tilde{\mathfrak{U}}$ is twisting with twisting family \mathfrak{F}_{MLL_u} . This means that we can represent any crossing of strings labeled by MLL_u formulas and these crossings interact as we attend with 2-cells which are not connected to control strings.

Remark 53 (Cut-gates shape) In light of De Morgan’s laws given in Definition 8, the possible *Cut-gates* inputs/outputs are the following:

- $Cut_{A\wp B} : (A \wp B), R, L, (B^\perp \otimes A^\perp) \Rightarrow \square$;
- $Cut_{A\otimes B} : (A \otimes B), R, L, (B^\perp \wp A^\perp) \Rightarrow \square$;
- $Cut_\perp : \perp, R, L, 1 \Rightarrow \square$;
- $Cut_1 : 1, R, L, \perp \Rightarrow \square$.

In this setting we are able to prove that the sequentializability of a diagram depends only on its inputs and outputs. Moreover, we are able to characterize MLL_u provable sequents in terms of existence of proof diagrams with a specific type.

Theorem 3 (Controlled proof diagram correspondence in $\tilde{\mathfrak{U}}$) *A sequent Γ is derivable in MLL_u if and only if there is a proof diagram $\phi \in \tilde{\mathfrak{U}}$ such that ϕ has no input and output of the form L, Γ, R , that is*

$$\vdash_{MLL_u} \Gamma \Leftrightarrow \exists \phi \in \tilde{\mathfrak{U}} \text{ such that } \phi : \square \Rightarrow L, \Gamma, R.$$

Proof To prove the left-to-right implication \Rightarrow , as in Theorem 2, we remark that, if there is a diagram $\phi : \square \Rightarrow L, \Gamma, R$ with Γ sequent in MLL_u , so there is a diagram

$$\phi^\sigma = (\mathbf{id}_L, \hat{\phi}_\sigma, \mathbf{id}_R) \circ \phi : \square \Rightarrow L, \sigma(\Gamma), R$$

for any permutation $\sigma \in S_{|\Gamma|}$. Then we proceed by induction on the number of inference rules in a derivation $d(\Gamma)$ in MLL_u :

- If just one inference rule occurs in $d(\Gamma)$, then it is an Ax or a 1 , then $\Gamma = A, A^\perp$ and $\phi_{d(\Gamma)} = Ax_A : \square \Rightarrow L, A, A^\perp, R$ or $\Gamma = 1$ and $\phi_{d(\Gamma)} = 1 : \square \Rightarrow L, 1, R$;
- If $n + 1$ inference rules appear in $d(\Gamma)$, we consider the last one and we distinguish two cases in base of its arity:

- If it is an unary \wp and $\Gamma = \Gamma', A \wp B$, then, by inductive hypothesis, there is a diagram $\phi_{d(\Gamma', A, B)} : \square \Rightarrow L, \Gamma', A, B, R$ of the derivation $d(\Gamma', A, B)$ and

$$\phi_{d(\Gamma)} = (\mathbf{id}_{L, \Gamma'}, \wp_{A, B}, \mathbf{id}_R) \circ \phi_{d(\Gamma', A, B)} : \square \Rightarrow L, \Gamma, R;$$

- Similarly, if it is a unary \perp and $\Gamma = \Gamma', \perp$, then, by inductive hypothesis, there is a diagram $\phi_{\Gamma'} : \square \Rightarrow L, \Gamma', R$ and $\phi_{\Gamma} = (L, \perp, \mathbf{id}_{\Gamma'}, R) \circ \phi_{\Gamma'}$;
- If it is a binary \otimes and $\Gamma = \Delta, A \otimes B, \Delta'$, then, by inductive hypothesis, there are two diagrams $\phi_{d(\Delta, A)} : \square \Rightarrow L, \Delta, A, R$ and $\phi_{d(B, \Delta')} : \square \Rightarrow L, B, \Delta', R$ relative to the two derivations $d(\Delta, A)$ and $d(B, \Delta')$ with at most n inference rules. Therefore

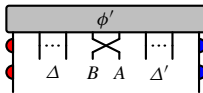
$$\phi_{d(\Gamma)} = (\mathbf{id}_{L, \Delta}, \otimes_{A, B}, \mathbf{id}_{\Delta', R}) \circ (\phi_{d(\Delta, A)}, \phi_{d(B, \Delta')}) : \square \Rightarrow L, \Gamma, R$$

- Similarly, if it is a binary *Cut* and $\Gamma = \Delta, \text{Cut}(A, A^\perp), \Delta'$, then

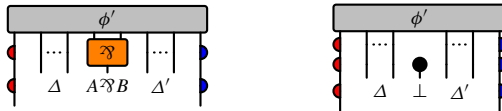
$$\phi_{d(\Gamma)} = (\mathbf{id}_{L, \Delta}, \text{Cut}_{A^\perp}, \mathbf{id}_{\Delta', R}) \circ (\phi_{d(\Delta, A)}, \phi_{d(A^\perp, \Delta')}) : \square \Rightarrow L, \Gamma, R.$$

In order to prove sequentialization, i.e. the right-to-left implication \Leftarrow , we proceed by induction on the number $|\phi|_S$ of gates in ϕ :

- If $|\phi|_{\tilde{\mathcal{U}}} = 0$ so $\phi : \mathbf{id}_{\Gamma} : \Gamma \Rightarrow \Gamma$. By hypothesis ϕ has no input (i.e. $s_2(\phi) = \square$) so it is the identity diagram over the empty string, this is the empty diagram $\mathbf{id}_0 : \square \Rightarrow \square$ which it is not sequentializable since $t_2(\phi) = \square \neq L, R$;
- If $|\phi|_{\tilde{\mathcal{U}}} = 1$ then ϕ is an elementary diagram. The elementary diagrams with source \square and target L, Γ, R with $\Gamma \in \tilde{\mathfrak{F}}_{\text{MLL}_u}^*$ are atomic made of a unique 2-cell of type $Ax_A : \square \rightarrow L, A, A^\perp, R$ for some $A \in \tilde{\mathfrak{F}}_{\text{MLL}_u}$ or $1 : 0 \rightarrow L, 1, R$. The associated sequent $\vdash A, A^\perp$ or $\vdash 1$ is derivable in MLL_u ;
- Otherwise there is 2-cell of type $\alpha : \Gamma' \Rightarrow \alpha(\Gamma') \in \tilde{\mathcal{U}}_2$ and $\Gamma = \Delta, \alpha(\Gamma'), \Delta'$. In this case $\phi = (\mathbf{id}_{L, \Delta}, \alpha, \mathbf{id}_{\Delta', R}) \circ \phi'$ where $\phi' : \square \Rightarrow L, \Delta, \Gamma', \Delta', R$. We have the following cases:
 - If $\alpha = \bowtie_{A, B}, \Gamma' = A, B$ and $\alpha(\Gamma') = B, A$. The diagram ϕ' is sequentializable by inductive hypothesis since $|\phi|_{\tilde{\mathcal{U}}} = |\phi'|_{\tilde{\mathcal{U}}} + 1$:



- Similarly if $\alpha = \wp_{A, B}, \Gamma' = A, B$ and $\alpha(\Gamma') = A \wp B$ or if $\alpha = \perp, \Gamma' = \emptyset$ and $\alpha(\Gamma') = \perp$:



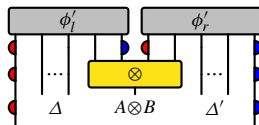
- If $\alpha = \otimes_{A, B}$ so $\Gamma' = A, R, L, B, \alpha(\Gamma') = A \otimes B$ and

$$\phi' : \square \Rightarrow L, \Delta, A, R, L, B, \Delta', R.$$

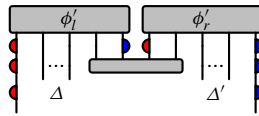
This diagram is a parallel composition $\phi = \phi'_l, \phi'_r$ with

$$\phi'_l : \square \Rightarrow L, \Delta, A, R \quad \text{and} \quad \phi'_r : \square \Rightarrow L, B, \Delta', R$$

of two diagrams which satisfy inductive hypothesis since $|\phi|_{\tilde{\mathcal{U}}} = |\phi'_l|_{\tilde{\mathcal{U}}} + |\phi'_r|_{\tilde{\mathcal{U}}} + 1$:



– Similarly if $\alpha = Cut_A$ with $B = A^\perp$ we have $\Gamma' = A, R, L, A^\perp$ and $\alpha(\Gamma') = \emptyset$:



□

In particular, this theorem gives a *representation procedure* to associate a diagram to a derivation and a *sequentialization procedure* to associate a derivation to a proof diagram. From the theorem we can deduce the complexity of a sequentializability test for proof diagrams.

Corollary 1 *The sequentializability of a proof diagram $\phi \in \tilde{\mathfrak{U}}$ can be checked in linear time on the number of inputs (if $\text{in}(\phi) = \square$) and outputs (if $\text{out}(\phi) = L, \Gamma, R$ with $\Gamma \in \mathfrak{F}_{\text{MLL}_U}$).*

In particular, this test does not depend on the number of gates in ϕ – and then the number of inference rules in the associate derivation – which is greater the number of conclusion formula only in case of a single-gate diagram – i.e. a single-rule derivation.

Definition 13 (Representation) We say that a proof diagram $\phi \in \tilde{\mathfrak{U}}$ with $\phi : \square \Rightarrow L, \Gamma, R$ represents a derivation $d(\Gamma)$ if it can be sequentialized into the derivation $d(\Gamma)$, and that a derivation $d(\Gamma)$ is represented by ϕ or that ϕ is a *diagrammatic representation* of $d(\Gamma)$ if the derivation $d(\Gamma)$ can be imitated by ϕ .

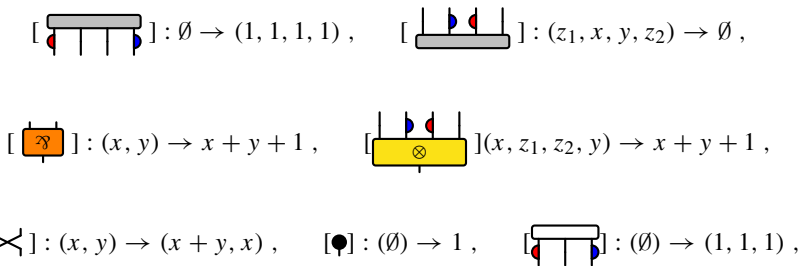
Definition 14 (Proof diagram branch) We say that ψ is a *branch* of a sequentializable proof diagram ϕ if it is a subdiagram of the form $\psi : \square \Rightarrow L, \Gamma, R$.

A branch $\psi \subseteq \phi$ represents to a sub-derivation of the derivation represented by ϕ , in other words it is a branch of the relative derivation tree.

We prove the termination of the polygraph $\tilde{\mathfrak{U}}$ in order to give a definition of *irreducible proof diagram*.

Theorem 4 (Termination of $\tilde{\mathfrak{U}}$) *The polygraph $\tilde{\mathfrak{U}}$ is terminating.*

Proof We define a *termination order* as in [13, 20] to prove termination by associating to any proof diagram $\phi : \Gamma \Rightarrow \Delta$ a function $[-] : \mathbb{N}^{|\Gamma|} \rightarrow \mathbb{N}^{|\Delta|}$ defined according to the following interpretations of gates:



In particular, for any rule $\phi \Rightarrow \phi' \in \tilde{\mathfrak{U}}_3$ we have such that $[\phi] > [\phi']$:

$$[\text{crossing}](x, y) = (2x + y, x + y) > (x, y) = [\text{gate}](x, y),$$

$$\begin{aligned}
 & \left[\text{Diagram 1} \right] (x, y, z) = (2x + y + z, x + y, x) > (x + y + z, x + y, x) = \left[\text{Diagram 2} \right] (x, y, z), \\
 & \left[\text{Diagram 3} \right] \emptyset = (0, 2, 1, 0) > (0, 1, 1, 0) = \left[\text{Diagram 4} \right] \emptyset, \\
 & \left[\text{Diagram 5} \right] (x, y, z) = (x + y + z + 1, x + y + 1) > (x + y + z, x + y + 1) = \left[\text{Diagram 6} \right] (x, y, z), \\
 & \left[\text{Diagram 7} \right] (x, y, z) = (2x + y + z + 1, x) > (x + y + z + 1, x) = \left[\text{Diagram 8} \right] (x, y, z), \\
 & \left[\text{Diagram 9} \right] (x) = (x + 1, 1) > (x, 1) = \left[\text{Diagram 10} \right] (x), \\
 & \left[\text{Diagram 11} \right] (x) = (x + 1, x) > (1, x) = \left[\text{Diagram 12} \right] (x),
 \end{aligned}$$

The compatibility of the order with sequential and parallel composition suffices to conclude that for any couple of diagrams $[\phi] > [\phi']$ holds whenever $\phi \xrightarrow{*} \psi$. This rules out the existence of an infinite reduction path by the same argumentations given in Theorem 1 proof. \square

In the next section we study the quotient over derivations induced by the morphisms in $\langle \tilde{\mathcal{U}} \rangle$.

5.3 The Quotient Over Derivations Induced by $\tilde{\mathcal{U}}$

The polygraph $\tilde{\mathcal{U}}$ generates a monoidal category $\langle \tilde{\mathcal{U}} \rangle$ where morphisms are the equivalence classes of proof diagrams generated by the signature \mathcal{U}_2 modulo the rewriting rules in \mathcal{U}_3 . The representability of a derivation by means of a proof diagram gives rise to an important question about the correlation between two derivations represented by the same proof diagram.

In this section we study the equivalence relation between derivations which can be represented by the same proof diagrams and by proof diagrams belonging to the same equivalence class in $\langle \tilde{\mathcal{U}} \rangle$. We compare it with the standard equivalence relation \sim and the equivalence relation induced over derivation by the proof net syntax [24].

If we denote $N_{d(\Gamma)}$ the proof net representing the derivation $d(\Gamma)$ and $\phi_{d(\Gamma)} \in \tilde{\mathcal{U}}$ a proof diagram representing a derivation $d(\Gamma)$, we can define the following equivalence relations over MLL_{\square} derivations:

- we denote \sim_N the equivalence relation over derivations induced by proof nets syntax. It is defined as follows:

$$d'(\Gamma) \sim_N d''(\Gamma) \text{ iff } N_{d'(\Gamma)} = N_{d''(\Gamma)}.$$

In other words, $d'(\Gamma) \sim_N d''(\Gamma)$ if and only if they can be represented by the same proof net.

- we denote \simeq_D the equivalence relation over derivations induced by proof diagram syntax. It is defined as follows:

$$d'(\Gamma) \simeq_D d''(\Gamma) \text{ iff } \exists \phi \in \tilde{\mathfrak{U}} \text{ such that } \phi_{d'(\Gamma)} = \phi = \phi_{d''(\Gamma)}.$$

In other words, $d'(\Gamma) \simeq_D d''(\Gamma)$ if and only if they can be represented by the same proof diagram in $\tilde{\mathfrak{U}}_3$.

- we denote $\sim_{\bar{D}}$ the equivalence relation over derivations induced by $(\tilde{\mathfrak{U}})$. It is defined as follows:

$$d'(\Gamma) \sim_{\bar{D}} d''(\Gamma) \text{ iff } \exists \phi_{d'(\Gamma)}, \phi_{d''(\Gamma)} \in \tilde{\mathfrak{U}} \text{ s.t. } [\phi_{d'(\Gamma)}]_{\tilde{\mathfrak{U}}} = [\phi_{d''(\Gamma)}]_{\tilde{\mathfrak{U}}}.$$

In other words, $d'(\Gamma) \sim_{\bar{D}} d''(\Gamma)$ if and only if they can be represented by two proof diagrams which are equivalent modulo $\tilde{\mathfrak{U}}_3$.

It is well-known that \sim_N captures all permutation of multiplicative inference rules except the ones changing the jump assignment for a \perp cell. This implies that $\sim_N = \sim$ over the pure multiplicative fragment of linear logic but that \sim_N is finer than \sim in presence of multiplicative units [16].

We remark that $\sim_{\bar{D}}$ captures all commutations of unary inference rules (\perp , \wp and exchange) with disjoint sets of principal and active formula occurrences (by the interchange rule) together with permutations between \perp or \wp rules and exchange rules (by twisting relations).

At the same time, in MLL_{\cup} sequent calculus we usually consider sequents as multisets; thus, the equivalence relation \simeq_D does not really take into account the geometry of twisting operators in proof diagrams. In fact, we can always re-arrange the order of occurrences of formulas in the sequents inside a derivation before representing it by a proof diagram. This allows to shape at will the geometry of twisting operators of the representation of the derivation. For this reason, unexpectedly (but not that much) it emerges that the two equivalence relations \simeq_D and $\sim_{\bar{D}}$ are equivalent.

However, this equivalence relation \simeq_D is not able to capture all permutations of binary inference rules (\otimes and *Cut*): let $\alpha, \beta \in \{\otimes, \text{Cut}\}$, then \sim equates only permutations of the kind that follows:

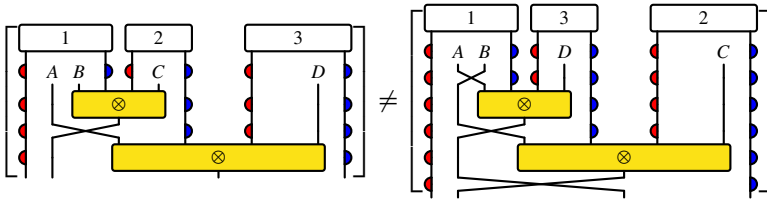
$$\frac{\frac{\frac{\vdots^1 \quad \vdots^2}{\vdash \Sigma, A \quad \vdash B, \Gamma, C} \alpha \quad \vdots^3}{\vdash \Sigma, \alpha(A, B), \Gamma, C} \beta}{\vdash \Sigma, \alpha(A, B), \Gamma, \beta(C, D), \Delta} \beta \quad \sim \quad \frac{\frac{\vdots^1 \quad \vdots^2 \quad \vdots^3}{\vdash \Sigma, A \quad \vdash B, \Gamma, C \quad \vdash D, \Delta} \beta}{\vdash \Sigma, \alpha(A, B), \Gamma, \beta(C, D), \Delta} \alpha$$

that is, permutations of \otimes or *Cut* rules that do not change the order of the branching in a derivation tree.

For an actual example of these particular cases, consider the linear logic sequent $B \otimes C, A \otimes D$. This exhibits two different \sim -equivalent (but also \sim_N -equivalent) derivations which are not \simeq_D -equivalent:

$$\frac{\frac{\frac{\vdots^1 \quad \vdots^2}{\vdash A, B \quad \vdash C} \otimes \quad \vdots^3}{\vdash A, B \otimes C} \otimes \quad \vdash D}{\vdash A \otimes D, B \otimes C} \otimes \quad \sim \quad \frac{\frac{\frac{\vdots^1 \quad \vdots^3}{\vdash A, B \quad \vdash D} \otimes \quad \vdots^2}{\vdash A \otimes D, B} \otimes \quad \vdash C}{\vdash A \otimes D, B \otimes D} \otimes$$

in fact, their diagrammatic representations belong to two different equivalence classes in $\langle \tilde{\mathcal{U}} \rangle$:



It follows that \simeq_D equates less than \sim . Most of all, the equivalence relation \simeq_D does not capture the part of the semantical equivalence which is required in order to take into account the elimination of commutative cuts and, consequently, to have an equivalence relation compatible with the cut-elimination result.

In the next section, we extend our polygraph in order to make compatible with cut-elimination the induced equivalence relation over derivations.

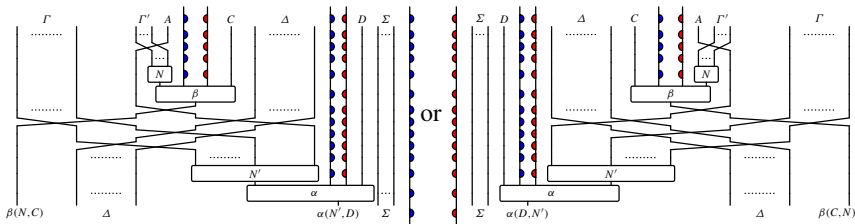
Remark 54 The two equivalences \simeq_D and \sim_N are not comparable. In fact, we have that \simeq_D captures \perp rules permutations which change jump assignments that are not captured by \sim_N , but \simeq_D does not capture permutations of binary inference rules which are perfectly captured by \sim_N .

5.4 The Polygraph of MLL_u Proof Diagrams

In this section we extend $\tilde{\mathcal{U}}$ to a polygraph \mathcal{U} in order to induce an equivalence over proof diagrams which captures the standard equivalence over derivations. To this end, we extend $\tilde{\mathcal{U}}$ with generators and rewriting rules in order to enable some permutations of proof diagram branches. In effect, these transformations are forbidden in $\tilde{\mathcal{U}}$ by the presence of control strings which impeach the definition of several twisting operators.

As remarked in the previous section, proof diagram syntax is inefficient to capture the standard proof equivalence in presence of some configurations including the ones of pure commutative cuts. This is because we keep records of how we manage occurrences of formulas in derivations (by means of twisting operators) revealing an hidden “tangle” structure.

Definition 15 (*Crossing split*) If $\phi \in \tilde{\mathcal{E}}$ is an irreducible proof diagram, we says that ϕ has a *crossing split* if it contains a subdiagram of the form



where α, β are *splitting gates*, that are gates of type \otimes or *Cut*, and $N, N' \in \{\text{cut}, \text{perp}, \perp\}^*$.

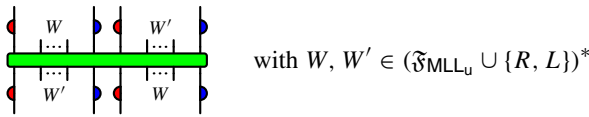
In other words, we have a crossing split in a proof diagram whenever the corresponding derivation exhibits two binary inference rules α after β such that the left (resp. right) active formula of α derives by the rightmost (resp. leftmost) sub-derivation branch of the left (resp. right) branch of β . For example, consider the two following configurations with A active formula of α :

$$\frac{\frac{\frac{\Gamma_A \quad \vdash \Gamma, \Gamma'_A, B \quad \vdash C, \Delta}{\vdash \Gamma, \Delta, \Gamma'_A, \beta(B, C)} \beta(B, C)}{\vdash \Gamma', \Delta', \Gamma'_A, \beta(B, C)} \quad \vdash D, \Sigma \quad \alpha(A, D)}{\vdash \Gamma', \Delta', \beta(B, C), \alpha(\Gamma'_A, D), \Sigma} \quad \text{OR} \quad \frac{\frac{\frac{\Gamma_A \quad \vdash \Delta, C \quad \vdash \Gamma, \Gamma'_A, B}{\vdash \Delta, \beta(B, C), \Gamma'_A, \Gamma} \beta(B, C)}{\vdash \Sigma, D \quad \vdash \beta(B, C), \Gamma'_A, \Gamma'} \alpha(D, A)}{\vdash \Sigma, \beta(B, C), \alpha(D, \Gamma'_A), \Gamma' \Delta'} \alpha(D, A)$$

where Γ_A and Γ'_A are sequents made of subformulas of A only (similarly for the formula B and Γ_B).

These configurations can be avoided in a proof diagram by giving a specific order to Ax and \perp -gates, in the same way we permute branches in derivation trees by \sim .

We call *untangle procedure* the method of removing crossing split from a proof diagram. This requires to perform some rewritings which permute proof diagram branches. For this purpose, we define some gates type with the following shape:



These gates can be seen as some “big twisting operators” able to cross a two sheafs of strings labeled by L, W, R and L, W', R where W, W' are lists made not only by formulas but also by L and R .

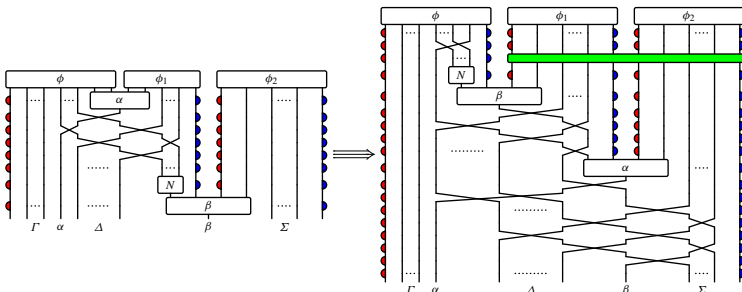
Definition 16 (*Polygraph of MLL_U*) The *polygraph of multiplicative proof diagrams* is the polygraph \mathfrak{U} obtained extended the polygraph $\tilde{\mathfrak{U}}$ as follows:

- $\mathfrak{U}_0 = \tilde{\mathfrak{U}}_0$;
- $\mathfrak{U}_1 = \tilde{\mathfrak{U}}_1$;
- $\mathfrak{U}_2 = \tilde{\mathfrak{U}}_2 \cup \mathfrak{B}ig$ where

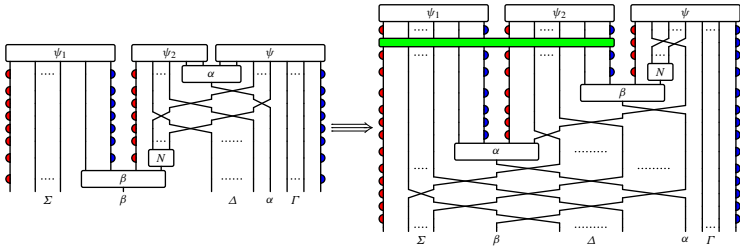
$$\mathfrak{B}ig = \left\{ B_{W, W'} = \begin{array}{c} \text{[Diagram of gate with green bar and strings]} \\ \text{with } W, W' \in (\mathfrak{F}_{\text{MLL}_U} \cup \{R, L\})^* \end{array} \right\};$$

- $\mathfrak{U}_3 = \tilde{\mathfrak{U}}_3 \cup \mathfrak{U}_{\mathfrak{B}ig}$ where $\mathfrak{U}_{\mathfrak{B}ig}$ is made of the following sets of 3-cells:

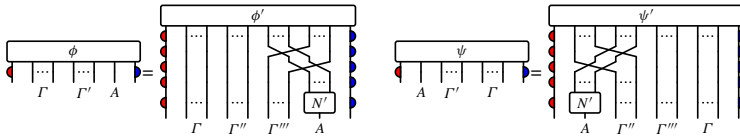
- \mathfrak{B} -introduction: for any $\alpha, \beta \in \{Cut, \otimes\}$ and $\phi, \phi_1, \phi_2, \psi, \psi_1, \psi_2, N, N'$ irreducible in $\tilde{\mathfrak{U}}$, with $N, N' \in \{\times, \wp, \perp\}^*$, we define the following 3-cells:



and

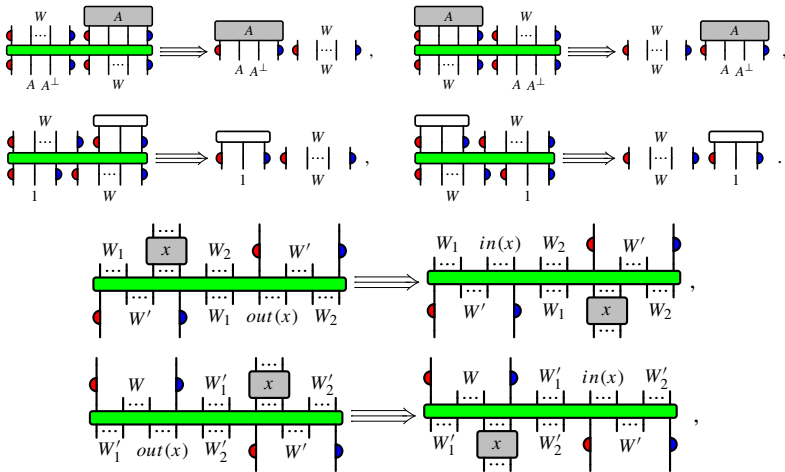


where ϕ and ψ are respectively of the form:



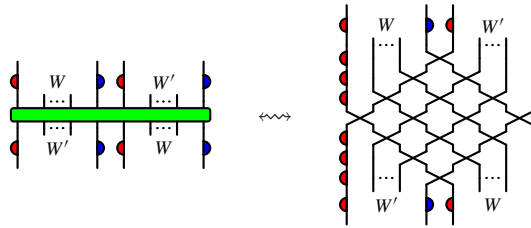
with $\Gamma' = \Gamma'', \Gamma'''$;

- The untangle relations: for any $A \in \mathfrak{F}_{MLL_U}$, $\begin{matrix} in(x) \\ \boxed{x} \\ out(x) \end{matrix} \in \tilde{\mathcal{E}}_2$, $B_{W,W'} \in \mathfrak{B}ig$ we define the following 3-cells:



where $W_1, in(x), W_2 = L, W, R$ and $W'_1, in(x), W'_2 = L, W', R$ with $W_1, W_2 \neq \square$ and $W'_1, W'_2 \neq \square$

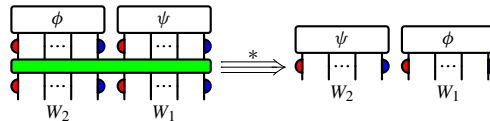
To have an intuition, a B -gate can be visualized as follows (but we remind the reader that such diagrams can not be defined in our syntax since twisting operators are not defined for control strings):



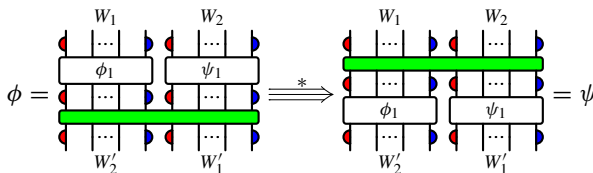
A \mathfrak{B} -introduction rule eliminates from a $\tilde{\Lambda}_3$ -irreducible proof diagram a crossing split: it exchanges the order of splitting gate, it modifies some twisting operators and it triggers the crossing of two proof diagram branches by the introduction of a B -gate.

At the same time, the untangle relations move gates from the top to the bottom of a B -gate according with our intuition: when a gate “crosses” a B -gate, it slides on the sheaf of strings passing from the left to the right and vice versa. These rules untangle, step-by-step, two crossed branches of a diagram:

Proposition 3 (*B-gate elimination*) *If $\phi, \psi \in \tilde{\Lambda}$ are proof diagrams of type $\phi : \square \Rightarrow L, W_1, R$ and $\psi : \square \Rightarrow L, W_2, R$ with $W_1, W'_2 \in (\mathfrak{F}_{\text{MLL}_u} \cup \{R, L\})^*$, then there are rewritings path made only of untangle relations of the following forms for gates of type B_{W_1, W_2} :*



We call this rewriting path a B-gate elimination. Moreover, if $\phi', \psi' \in \tilde{\Lambda}$ are proof diagrams of type $\phi' : W_1 \Rightarrow L, W'_1, R$ and $\psi' : W_2 \Rightarrow L, W'_2, R$ with $W_1, W'_1, W_2, W'_2 \in (\mathfrak{F}_{\text{MLL}_u} \cup \{R, L\})^$, then there are rewritings path made only of untangle relations of the following forms:*

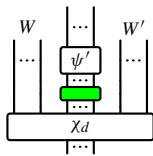


We call this rewriting path a B-gate reduction.

Proof By induction over the number of gates in the diagram ϕ, ψ : each untangle relation decrease it. □

We call *untangle sequence* a rewriting path made of one \mathfrak{B} -introduction rule followed by its relative B -gate elimination rewriting path. Each untangle sequence corresponds to the elimination of a crossing split and it terminates after a finite number of steps depending on the number of gates in the diagram.

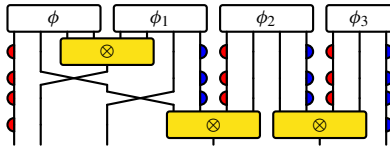
We have a maximal B -gate reduction $\phi \xrightarrow{*} \phi'$ when ϕ' is of the form:

$$\phi' = \chi_d \circ (\mathbf{id}_W, (\psi' \circ B), \mathbf{id}_{W'}) =$$


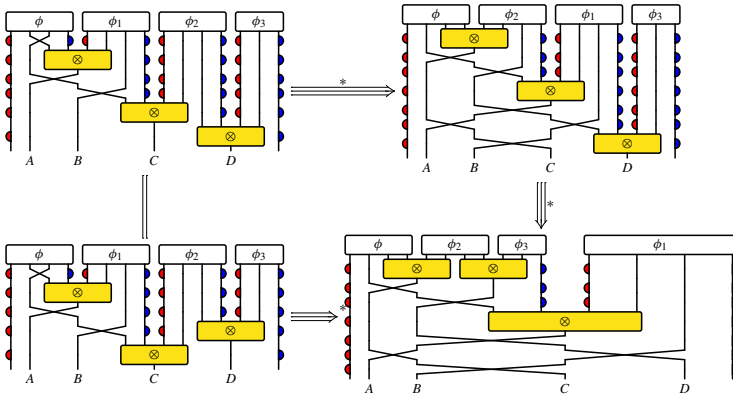
$$\text{with } \psi' \in \mathfrak{B}ig^*$$

We call any such rewriting path a B -deactivation.

We assume that any of this sequence generate no new crossing split. In fact, the elimination of a crossing split generate a new one if and only if there is a gate corresponding to a binary inference rule in parallel with respect of the lower splitting gate, for example:



In these cases it is possible to verify that either we apply the \mathfrak{B} -introduction rule in such a way as to maintain these two gates in the same branching of the diagram, or we perform a second untangle sequence we are able to recover a configuration where they are in parallel again. In the previous example we have:



The choice of define \mathfrak{B} -introduction rules with premises $\tilde{\mathfrak{U}}_3$ -irreducible diagrams with no B -gates leads the following result:

Corollary 2 *Conflicts between a \mathfrak{B} -introduction rule and an untangle relation and conflicts between a rule in \mathfrak{U}_{Big} and a rule in $\tilde{\mathfrak{U}}_3$ are trivially solvable. Then we can assume the corresponding rewritings paths commute.*

Proof The subdiagram rewritten by a \mathfrak{B} -introduction rules is $\tilde{\mathfrak{U}}_3$ -irreducible and contains no B -gates then all possible non-trivial conflicts are the ones between two \mathfrak{B} -introduction rules discussed above. The confluence of non-trivial critical peaks between untangle relations and rules in $\tilde{\mathfrak{U}}_3$ follows by argumentations similar to the ones given in the Proposition 3. \square

This lead the following theorem about the termination of rewriting in \mathfrak{U} .

Theorem 5 *(Termination in \mathfrak{U}) The polygraph \mathfrak{U} is terminating.*

Proof Corollary 2 implies that a rewriting path in \mathfrak{U} can be written as an alternate sequence of rewriting paths in \mathfrak{U}_3 , untangle sequences and B -deactivations. We know that the length of untangle sequences and B -deactivations are finite and linearly depends on the number of gates in a diagram. Moreover, Theorem 4 proves that there are not infinite rewriting paths composed of rules in \mathfrak{U}_3 . Then, to prove termination it suffices to prove that the number n of alternations is finite.

For any $\phi \in \mathfrak{U}$, if ϕ_{Cross} is the number of crossing splits in ϕ , any alternate rewriting path starting from ϕ counts at most $|\phi|_{\mathfrak{B}ig}$ B -deactivations and ϕ_{Cross} untangle sequences. In fact, no rule in $\tilde{\mathfrak{U}}_3$ generates new B -gates either crossing splits. This is underlined by the correspondence between equivalence relation over derivations $\simeq_D = \sim_{\tilde{D}}$ and rules permutations over derivations which do not change the structure of tree branching. \square

We extend the Theorem 3 to proof diagrams in $\tilde{\mathfrak{U}}$. This leads the linear complexity of the test of sequentializability for proof diagrams in \mathfrak{U} .

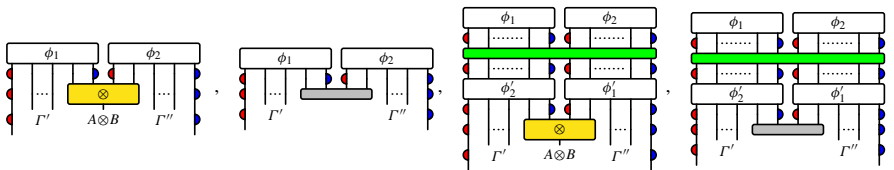
Theorem 6 (Multiplicative proof diagram correspondence)

$$\vdash_{\text{MLL}_{\mathfrak{U}}} \Gamma \Leftrightarrow \exists \phi \in \mathfrak{U} \text{ such that } \phi : \square \Rightarrow L, \Gamma, R.$$

Proof The left-to-right implication immediate follows by Theorem 3. For the proof of right-to-left implication we have to also consider the cases when it occurs a 2-cells in $\mathfrak{B}ig$. We observe that a proof diagram $\phi : \square \Rightarrow L, \Gamma, R$ contains a gate of type $B \in \mathfrak{B}ig$ iff there is a subdiagram $\phi' \subseteq \phi$ of the form

$$\phi' = (\mathbf{id}_{L, \Gamma'}, g_{\alpha}, \mathbf{id}_{\Gamma'', R}) \circ (\phi'_2, \phi'_1) \circ B \circ (\phi_1, \phi_2)$$

with g_{α} gate of type $\alpha \in \{Cut, \otimes\}$. Then, during the sequentialization procedure, whenever a gate of type \otimes or Cut occurs, we consider the following cases:



the first two cases are handled by the same strategy of Theorem 3. The sequentialization procedure for the two new cases follows the intuition behind B -gates as proof diagram branchings twisting: $(\mathbf{id}_{L, \Gamma'}, g_{\alpha}, \mathbf{id}_{\Gamma'', R}) \circ (\phi'_2, \phi'_1) \circ B \circ (\phi_1, \phi_2) : \square \Rightarrow L, \Gamma'', R, L, \Gamma', R$ is sequentializable iff $\phi'_1 \circ \phi_1 : \square \Rightarrow L, \Gamma', R$ and $\phi'_2 \circ \phi_2 : \square \Rightarrow L, \Gamma'', R$ are. \square

Remark 55 The signature $\tilde{\mathfrak{U}}_2$ suffice to represent $\text{MLL}_{\mathfrak{U}}$ derivations, that is, B -gates are not needed in order to represent proofs and that the quotient $\langle \mathfrak{U} \rangle$ equate more of these proof diagrams than $\langle \tilde{\mathfrak{U}} \rangle$.

Let consider the equivalence relation \sim_D over derivations of $\text{MLL}_{\mathfrak{U}}$ sequent calculus defined as follows:

$$d'(\Gamma) \sim_D d''(\Gamma) \text{ iff } \exists \phi_{d'(\Gamma)}, \phi_{d''(\Gamma)} \in \tilde{\mathfrak{U}} \text{ such that } [\phi_{d'(\Gamma)}]_{\mathfrak{U}} = [\phi_{d''(\Gamma)}]_{\mathfrak{U}}.$$

where $\phi_{d(\Gamma)} \in \mathfrak{U}$ is a diagrammatic representation of a $\text{MLL}_{\mathfrak{U}}$ derivation $d(\Gamma)$. In other words, $d'(\Gamma) \sim_D d''(\Gamma)$ whenever they can be represented by two proof diagrams which are equivalent modulo \mathfrak{U}_3 .

The standard proof equivalence of $\text{MLL}_{\mathfrak{U}}$ sequents is faithfully represented by \sim_D :

Theorem 7 (Proof diagram representation) *Two derivations are equivalent modulo \sim if and only if they are represented by two equivalent proof diagrams with respect of $\langle \mathfrak{U} \rangle$. That is:*

$$d(\Gamma) \sim d'(\Gamma) \Leftrightarrow d(\Gamma) \sim_D d'(\Gamma)$$

Proof Given two derivation $d(\Gamma), d'(\Gamma)$ in MLL_u sequent calculus, $d(\Gamma) \sim d'(\Gamma)$ iff there is a sequence of rules permutations from $d(\Gamma)$ to $d'(\Gamma)$. As remarked in Sect. 5.3, $\sim_{\bar{D}}$ capture all rules permutations which do not affect the branching of a derivation tree and $\simeq_D \subset \sim_D$.

This implies that even if we consider derivations up to rules permutations, it is possible to well-define the following function which associate to a derivation an equivalence class of proof diagrams in \mathfrak{U} :

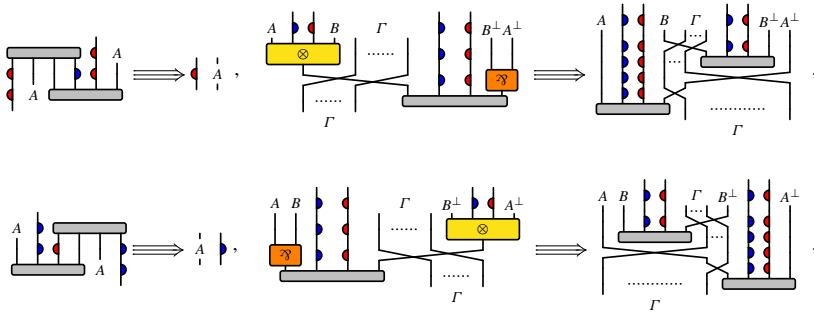
$$\begin{aligned} [-]_{\mathfrak{U}} : \{ \text{MLL}_u \text{ derivations} \} &\rightarrow \{ \text{morphisms in } \langle \mathfrak{U} \rangle \} \\ d(\Gamma) &\rightarrow [\phi_{d(\Gamma)}]_{\mathfrak{U}} \end{aligned}$$

Moreover, in a diagrammatic representation of a derivation (Γ) , untangle sequences and their inverses permute pairs of proof diagram branches which correspond to the represented derivation branches. This means that \sim_D captures all rules permutations missed by \simeq_D , then that $\sim = \sim_D$ □

We define the following polygraph:

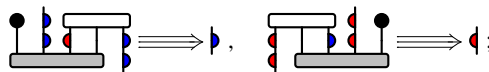
Definition 17 (Polygraph of MLL_u semantics) *The polygraph of multiplicative linear logic semantics S_{MLL_u} is given by extending the polygraph \mathfrak{U} with the following the sets of 3-cells $S_3 = \mathfrak{U}_3 \cup S_{\text{MLL}_u}^{\text{Cut}}$ where $S_{\text{MLL}_u}^{\text{Cut}} = \mathfrak{M}_{\text{Cut}} \cup \mathfrak{U}_{\text{Cut}}$ is given by the following sets of 3-cells:*

- $\mathfrak{M}_{\text{Cut}}$ is made of the following 3-cells:



for all $A, B \in \mathfrak{F}_{\text{MLL}_u}, \Gamma \in \mathfrak{F}_{\text{MLL}_u}^*$;

- $\mathfrak{U}_{\text{Cut}}$ is made of the following 3-cells:



Theorem 8 (Termination in S_{MLL_u}) *The polygraph S_{MLL_u} is terminating.*

Proof Any rewriting path in S_3 is a sequence of rewriting paths in \mathfrak{U} and rewriting rules in $S_{\text{MLL}_u}^{\text{Cut}}$ occurrences. If the number these latter is finite in any rewriting path, we conclude by Theorem 5 that there are no infinite rewriting paths in S_{MLL_u} .

We define the *degree* $\delta(g) = \|A\|$ of Cut_A -gates $g \in \phi$ as the number of occurrences of \wp and \otimes symbols in the formula A . We define a weight $w(\phi)$ of a proof diagram $\phi \in \mathfrak{U}$ depending on the degrees of all its Cut -gates:

$$w(\phi) = \sum_{g:Cut}^{g \in \phi} 3^{\delta(g)}$$

We observe that $w(\phi) = w(\psi)$ whenever $\phi \xrightarrow{\mathfrak{U}_3} \psi$ since ϕ and ψ have the same occurrences of Cut -gates.

However, $w(\phi) > w(\psi)$ whenever $\phi \xrightarrow{\mathfrak{S}_3} \psi$. In fact, ϕ has an extra Cut -gates with respect to the one of ψ or else in ϕ there is a $Cut_{A \otimes B}$ -gate or a $Cut_{A \wp B}$ -gate which is replaced in ψ by one Cut_A -gate and one Cut_B -gate. The inequality holds because for any $A, B \in \mathfrak{F}_{MLL_{\perp}}$ we have $3^{A \otimes B} = 3^{A \wp B} = 3^{\|A\| + \|B\| + 1} > 3^{\|A\|} + 3^{\|B\|}$,

This concludes the proof since any rewriting path in \mathfrak{S}_3 there is a finite number of occurrence rewriting rules in $\mathfrak{S}_{MLL_{\perp}}^{Cut}$ □

Consequently, we have a *cut-elimination* Theorem for sequentializable proof diagrams in $\mathfrak{S}_{MLL_{\perp}}$

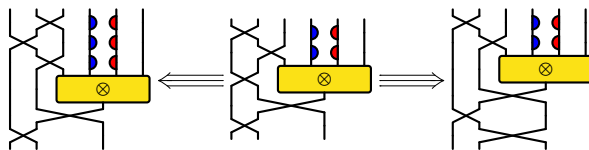
Theorem 9 (Cut-elimination) *An irreducible proof diagram $\phi \in \mathfrak{S}_{MLL_{\perp}}$ which represent a derivation contains no Cut -gates.*

Proof Proposition 3 assures that a \mathfrak{U}_3 -irreducible proof diagram $\phi \in \mathfrak{S}_{MLL_{\perp}}$ of type $\phi : \square \Rightarrow L, \Gamma, R$ contains no B -gates and, by Theorem 8, neither crossing splits. Since twisting relations moves \wp and \perp gates downward in a proof diagram ϕ , if a Cut_A -gate occurs in ϕ then it has to belong in a subdiagram with shape the source one of the rules in \mathfrak{S}_3 . □

However, the twisting relations generate a wide family of critical peaks in the rewritings of $\tilde{\mathfrak{U}}$, \mathfrak{U} and $\mathfrak{S}_{MLL_{\perp}}$. Some of these critical peaks are not solvable. This leads the following:

Proposition 4 ($\mathfrak{S}_{MLL_{\perp}}$ confluence) *The polygraph $\mathfrak{S}_{MLL_{\perp}}$ is not confluent.*

Proof In $\mathfrak{S}_{MLL_{\perp}}$ (but also in $\tilde{\mathfrak{U}}$ and \mathfrak{U}) the following critical peak is not confluent:



This rules out a confluence for this polygraph. □

Since the signature of $\mathfrak{S}_{MLL_{\perp}}$ is the same as that of MLL_{\perp} , we naturally extend the Theorem 6:

Theorem 10 (Multiplicative linear logic correspondence)

$$\vdash_{MLL_{\perp}} \Gamma \Leftrightarrow \exists \phi \in \mathfrak{S}_{MLL_{\perp}} \text{ such that } \phi : \square \Rightarrow L, \Gamma, R.$$

After Theorem 7, this correspondence ensures the well-definition of the following function:

Definition 18 (*Denotational semantics of proof diagrams*) For any MLL_U derivation $d(\Gamma)$ we associate an equivalence class of proof diagrams, that is a morphism of the category $\langle \mathcal{S}_{\text{MLL}_U} \rangle$ generated by $\mathcal{S}_{\text{MLL}_U}$ as follows:

$$[-]_D : \{ \text{MLL}_U \text{ derivations} \} \rightarrow \{ \text{morphisms in } \langle \mathcal{S}_{\text{MLL}_U} \rangle \}$$

$$d(\Gamma) \rightarrow [d(\Gamma)]_D = [\phi_{d(\Gamma)}]_{\mathcal{S}_{\text{MLL}_U}}$$

where $\phi_{d(\Gamma)}$ is an arbitrary representation of $d(\Gamma)$.

If $d(\Gamma, A)$ and $d(A^\perp, \Delta)$ are two derivations, we define the *semantical composition* of $[d(\Gamma, A)]_D$ and $[d(A^\perp, \Delta)]_D$ (denoted \bullet) as the equivalence class $[d(\Gamma, A)]_D \bullet [d(A^\perp, \Delta)]_D = [(\mathbf{id}_{L,\Gamma}, \text{cut}_A, \mathbf{id}_{\Delta,R}) \circ (\phi_{d(\Gamma,A)}, \phi_{d(A^\perp,\Delta)})]_D$.

We remark that even if $\mathcal{S}_{\text{MLL}_U}$ has the structure of monoidal category, its product (the parallel composition) is not compatible with the map $[-]_D$. In fact, the parallel composition of two sequentializable proof diagrams is not sequentializable (it corresponds to the *mix* rule, the binary rule with conclusion the juxtaposition of the two premise sequents). Then $([-]_D, \bullet)$ is not a substructure of $\mathcal{S}_{\text{MLL}_U}$, but it is a category with objects the images of $[-]_D$ in the category $\mathcal{S}_{\text{MLL}_U}$.

Theorem 11 (Proof diagram semantics) $([-]_D, \bullet)$ is a denotational semantics for MLL_U sequent calculus.

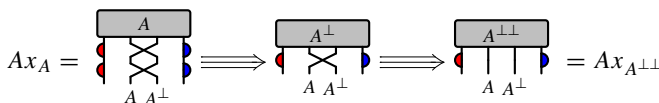
Proof We define the following equivalence relation \approx_D over MLL_U derivations:

$$d'(\Gamma) \approx_D d''(\Gamma) \text{ if and only if } [d'(\Gamma)]_D = [d''(\Gamma)]_D$$

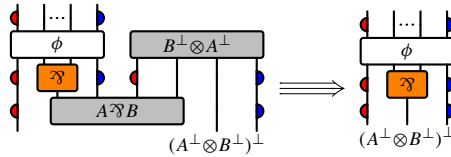
There exists a one-to-one correspondence between rewriting rules in $\mathcal{S}_{\text{MLL}_U}^{\text{Cut}}$ and cut-elimination steps. Moreover, if we denote by $\leftrightarrow_{\text{Cut}}^*$ the equivalence relation induced over equivalence classes in $\langle \mathcal{U} \rangle$ by the rewriting rules in $\mathcal{S}_{\text{MLL}_U}^{\text{Cut}}$, we have that $\langle \mathcal{S}_{\text{MLL}_U} \rangle = \frac{\langle \mathcal{U} \rangle}{\leftrightarrow_{\text{Cut}}^*}$. Thus, the following properties hold:

1. if $d(\Gamma) \rightarrow_{\text{Cut}} \hat{d}(\Gamma)$, then $d(\Gamma) \approx_D \hat{d}(\Gamma)$: each cut-elimination step over the diagrammatic representation of $\phi_d(\Gamma)$ is replicated by a rewriting rule in $\mathcal{S}_{\text{MLL}_U}^{\text{Cut}}$ eventually preceded by a rewriting path $\Rightarrow_{\mathcal{U}}^*$ in \mathcal{U}_3 .
2. \approx_D is non-degenerated, i.e. one can find a formula with at least two non-equivalent proofs: it suffices to take any formula $A \in \mathfrak{F}_{\text{MLL}_U}$ which exhibits two non-equivalent (with respect of \sim) cut-free derivations $d(A)$ and $d'(A)$, then trivially $d(A) \not\approx_D d'(A)$;
3. \approx_D is a congruence, i.e. if $d(\Delta) \approx d'(\Delta)$ and we obtain $d(\Gamma)$ and $d'(\Gamma)$ by applying the same inference rule to $d(\Delta)$ and $d'(\Delta)$, then, from the compatibility of rewriting with sequential and parallel diagram compositions, it follows that $d(\Gamma) \approx d'(\Gamma)$.

We remark that $[-]_D$ is coherent with the involutivity of negation. In fact, the invariance of diagram inputs and outputs with respect to rewriting imposes the equivalence $A^{\perp\perp} = A$:



Similarly, De Morgan’s laws follow by the definition of *Cut*-gates (see Remark 53). By means of example, consider the equivalence of $A \wp B = (B^\perp \otimes A^\perp)^\perp$:



From these properties we deduce that $[-]_D$ defines a denotational semantics for MLL_{\perp} sequent calculus by means of equivalence classes of proof diagrams with composition \bullet . \square

6 Conclusion and Future Work

In this paper we have presented the syntax of *proof diagrams*, a particular class of string diagrams suitable to represent linear logic proof derivations, and we have studied its properties. Even if this syntax reminds the intuitive 2-dimensional representations of proof nets, proof diagrams strings have a more rigid structure with respect to proof structures wirings. This allows for the definition of some *control strings* and a consequent sequentializability test. Indeed, in this setting we can test when a proof diagram can be interpreted as a MLL_{\perp} derivation in linear time by checking the type of its inputs and outputs only.

Furthermore, the syntax of proof diagrams induce an equivalence relation over the syntax of MLL_{\perp} sequent calculus derivations. In the paper we have studied some different equivalence relations over derivation induced by some rewriting systems defined over proof diagrams; we here summarize the principal results:

- proof diagrams syntax induces an equivalence relation over derivation capturing all permutations of \wp and \perp rules, but not the permutations involving *Cut* and \otimes that also permute derivation branches order. This equivalence turns out to be invariant under a given set of diagram rewriting rules we call *twisting relations*;
- we can simulate derivation branches permutations by enriching our representation with some additional gates and rewriting rules. This allows us to capture the standard proof equivalence of MLL_{\perp} sequent calculus;
- we can simulate cut-elimination steps by rewriting rules and prove cut-elimination theorem for proof diagrams independently from sequent calculus result.

These results allow us to define a denotational semantics for MLL_{\perp} sequent calculus by means of equivalence classes of proof diagrams generated by a polygraph $S_{MLL_{\perp}}$. For this semantics we define a composition of proofs which can be interpreted as the application of *Cut* between the conclusion formulas of the two proofs.

We showed that the diagram rewriting systems we have introduced are terminating. However, the presence of control strings added to guarantee a linear sequentializability test rules out the confluence of the rewriting systems. In conclusion, we can affirm that interchange law for two dimensional string diagrams does not suffice to capture syntactically the notion proof equivalence.

This work suggests several future research directions. In particular, in the near future, we will focus on extending the present results to the multiplicative-exponential linear logic fragment, and we will use the generalization of the interchange law in higher-dimensional categories to syntactically capture the whole proof equivalence without the use of rewriting systems.

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