Canonicity of Proofs in Constructive Modal Logic ⋆

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Abstract. In this paper we investigate the Curry-Howard correspondence for constructive modal logic in light of the gap between the proof equivalences enforced by the lambda calculi from the literature and by the recently defined winning strategies for this logic.
We define a new lambda-calculus for a minimal constructive modal logic by enriching the calculus from the literature with additional reduction rules and we prove normalization and confluence for our calculus. We then provide a typing system in the style of focused proof systems allowing us to provide a unique proof for each term in normal form, and we use this result to show a one-to-one correspondence between terms in normal form and winning innocent strategies.

Keywords: Constructive Modal Logic, Lambda Calculus, Game Semantics

1 Introduction

Proof theory is the branch of mathematical logic whose aim is studying the properties of logical arguments (i.e., proofs) as well as the structure of proofs and their invariants. For this purpose, the most used representations of proofs are based on tree-like data structures inductively defined using inference rules of a proof system.[4] *Natural deduction* and *sequent calculus* are among the most used proof systems due to their intuitive representation. Both these proof systems were originally devised by Gentzen in order to prove the consistency of first-order arithmetic. Their versatility resulted in their employment for a wide variety of logics.

However, having formalisms able to represent proofs is not enough to define "what is a proof" since different derivations, or derivations in different proof systems, could represent the same abstract object. A notion of *proof identity* is therefore required to define a proof as a proper mathematical entity [18]. Such a notion of identity is provided by delineating the conditions under which two distinct formal representations of a proof represent the same logical argument. The definition of these conditions are often driven

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[4] It is worth noting that some proof systems (in the sense of [12]) allows to represent proofs using structures such as infinite trees (for non-well-founded proof systems, see, e.g., [15]), graphs (see proof nets [22,23], combinatorial proofs [27,27] or proof diagrams [3]) or structures defined in a compositional way (see open deduction [24] and deep inference [50])

by semantic considerations (by performing specific transformations on two derivations, they can be transformed to the same object) or intuitive ones (two derivations only differ for the order in which the same rules are applied to the same formulas).

Natural deduction is often considered a satisfactory formalism since it allows to define a more canonical representation of proofs with respect to sequent calculus: sequent calculus derivations differing because of some rules permutations are represented (*via* a standard translation) by the same natural deduction derivation. Moreover, natural deduction provides a one-to-one correspondence between derivations and lambda-terms, called the *Curry-Howard correspondence* [48].

Constructive Modal Logic. Classical modal logics are obtained by extending *classical logic* with unary operators, called *modalities*, that qualify the truth of a judgment. The most used modalities are the $\Box$ (called *box*) and its dual operator $\Diamond$ (called *diamond*) which are usually interpreted as *necessity* and *possibility*. According to the interpretation of such modalities, modal logics find applications, for example, in knowledge representation [51], artificial intelligence [40] and the formal verification of computer programs [45,19,36]. The work of Fitch [21] initiated the investigation of the proof theory of modal logics extending intuitionistic logic, leading to numerous results on the topic [46,26,39,20,35].

In particular, the Curry-Howard correspondence has been extended to various constructive modal logics [6,9,31,32,44,16]. Intuitionistic logic can be extended with modalities in different ways (for an overview see [47]): while in classical logic axioms involving only $\Box$ provide also description of the behavior of $\Diamond$, for intuitionistic logic this is no more the case since the duality of the two modalities does not hold anymore. This leads to different approaches. *Constructive modal logics* consider minimal sets of axioms to guarantee the definition of the behaviors of the $\Box$ and $\Diamond$ modalities. A second approach, referred to as *intuitionistic modal logic*, considers additional axioms in order to validate the Gödel-Gentzen translation [14]. In this work we consider a minimal fragment of the constructive modal logic CK only containing the implication $\to$ and the modality $\Box$. This fragment is enough to define types for a λ-calculus with a Let constructor [6] which can be interpreted as an explicit substitution and, for this reason, we more concisely denote by $N\,[M_1, \dots M_n/x_1, \dots, x_n]_\blacksquare$ instead of $\mathsf{Let}\ M_1, \dots M_n\ \mathsf{be}\ x_1, \dots, x_n\ \mathsf{in}\ N$.

Recent works on the the proof equivalence of constructive modal logics [5] expose a complexity gap between the proof equivalences induced by the natural deduction ([9]) and winning innocent strategies ([4]) for this logic. This discrepancy cannot be observed in intuitionistic propositional logic where there are one-to-one correspondences between natural deduction derivations, lambda terms and innocent winning strategies. In particular, in the logic CK we observe sequent calculus proofs which correspond to the same winning strategy but which cannot be represented by the same natural deduction derivation in the systems provided in [9,31] (or equivalently corresponding to different modal λ-terms). By means of example, consider the terms $x\,[z/x]_\blacksquare$ and $x\,[z, w/x, y]_\blacksquare$ and their (unique) typing derivations shown in Figure 1 (see Figure 3 for the typing system). Intuitively, the two terms $x\,[z/x]_\blacksquare$ and $x\,[z, w/x, y]_\blacksquare$ should be semantically equivalent since the explicit substitution of the variable y in the term x is vacuous. Said differently, if we explicit the substitution encoded by the constructor Let, both terms $x\,[z/x]_\blacksquare$ and $x\,[z, w/x, y]_\blacksquare$ should reduce to the term z.

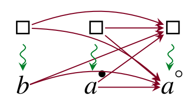
$$\Box\text{-subst}\ \dfrac{\text{Id}\ \dfrac{}{z : \Box a, w : \Box b \vdash z : \Box a} \qquad \text{Id}\ \dfrac{}{x : a, y : b \vdash x : a}}{z : \Box a, w : \Box b \vdash x\,[z/x]_{\blacksquare} : \Box a}$$

$$\Box\text{-subst}\ \dfrac{\text{Id}\ \dfrac{}{z : \Box a, w : \Box b \vdash z : \Box a} \qquad \text{Id}\ \dfrac{}{z : \Box a, w : \Box b \vdash w : \Box b} \qquad \text{Id}\ \dfrac{}{x : a, y : b \vdash x : a}}{z : \Box a, w : \Box b \vdash x\,[z, w/x, y]_{\blacksquare} : \Box a}$$

Fig. 1. The typing derivations of the modal λ-terms $x\,[z/x]_{\blacksquare}$ and $x\,[z, w/x, y]_{\blacksquare}$.

In fact, this undesirable behavior disappear when considering the Winning Innocent Strategies for CK defined in [4]. In this syntax, both the above natural deduction derivations correspond to the same strategy below.

$$\mathcal{S} = \{\epsilon, a^{\circ}, a^{\circ}a^{\bullet}\} \qquad \text{over the arena} \qquad [\![\Box a, \Box b \vdash \Box a]\!] \quad = \tag{1}$$



Contribution. In this paper we define a new modal λ-calculus for CK by considering additional rewriting rules that allow us to retrieve a one-to-one correspondence between terms in normal form and winning innocent strategies, that is, providing more canonical representatives for proofs with respect to natural deduction and modal λ-terms defined in the literature. From the technical point-of-view, we obtain this result by extending the operational semantics of the modal λ-calculus with the appropriate new reduction rules for the explicit substitution encoded by the Let, dealing with contraction and weakening operating on the variables bound by the Let. We call this set of rules the κ-reduction, which we show to be strongly normalizing using elementary combinatorial methods. In order to deal with the interaction of the η-reduction with β-reduction, we define a restricted η-reduction following an approach similar to the one used in [42,17,30]. We prove strong normalization and confluence for our new operational semantics.

After proving confluence and strong normalization for our modal λ-calculus, we provide a canonical typing system inspired by focused sequent calculi (see, e.g., [7]) providing a unique typing derivation for each term in normal form. We conclude by establishing a one-to-one correspondence between the winning strategies defined in [4] and proofs of this calculi, therefore with terms in normal form.

Related Work. To the best of our knowledge, the first paper proposing a Curry-Howard correspondence for the logic CK is [9]. In this work, the authors provide a natural deduction system for the logic CK by enriching the standard system for intuitionistic propositional logic with a generalized elimination rule capable of taking into account the behavior of the $\Box$-modality. At the level of lambda calculus, they enrich the syntax of terms by adding a new constructor Let defined as follows:

$$\text{Let } x_1, \ldots x_n \text{ be } N_1, \ldots, N_n \text{ in } M \qquad (\text{which we denote } M\,[N_1, \ldots, N_n/x_1, \ldots, x_n]_{\blacksquare}) \tag{2}$$

providing a notation which can be interpreted as an explicit substitution of the variable x_i with the term N_i for all occurrences of $x_1 \ldots, x_n$ inside a term M. For this calculus, the authors only consider the usual η and β reductions plus the following reduction:

$$\text{Let } y \text{ be } P \text{ in } (\text{Let } x \text{ be } N \text{ in } M\ ) \rightsquigarrow \text{Let } x \text{ be } (\text{Let } y \text{ be } P \text{ in } N\ ) \text{ in } (\text{Let } x \text{ be } x \text{ in } M\ )$$
$$\left(\text{in our syntax this reduction is written as} \quad M\,[N/x]_{\blacksquare}\,[P/y]_{\blacksquare} \rightsquigarrow M\,[x/x]_{\blacksquare}\,[N\,[P/y]_{\blacksquare}\,/x]_{\blacksquare}\right)$$

In [31] the author considers the usual η and β reduction with an the following additional β-reduction rule specifically designed to handle the explicit substitution construct.

$$M\left[\vec{P}, R\left[\vec{N}/\vec{z}\right]_{\blacksquare}, \vec{Q}/\vec{x}, y, \vec{w}\right]_{\blacksquare} \rightsquigarrow_{\beta_2} M\,\{R/y\}\left[\vec{P}, \vec{N}, \vec{Q}/\vec{x}, \vec{z}, \vec{w}\right]_{\blacksquare} \tag{3}$$

In the same paper, the author provides a detailed proof of strong normalization and confluence for modal lambda terms with respect to the standard η and β reduction, plus this new β_2 reduction. However, also this calculus does not manage to fix the aforementioned problem with canonicity.

An alternative natural deduction system (and λ-calculus) is proposed in [32], where the symmetry between elimination and introduction rules typical of natural deduction is restored. However, this result requires to define a sequent calculus where sequents have a more complex structure (dual-contexts), and lacks an in-depth study of the operational semantics because the η-expansion is not considered in the calculus.

Outline of the paper. In Section 2 we recall the definition of the fragment of the logic CK we consider in this paper, as well as the main results on the proof theory for this logic, its natural deduction and lambda calculus. In Section 3 we define the modal λ-calculus we consider in this paper, proving its strong normalization and confluence properties. In Section 4 we provide a typing system in the style of focused sequent calculi, where we are able to narrow the proof search of the type assignment of our normal terms to a single derivation. In Section 5 we recall the definition of the game semantics for the logic we consider and we prove the one-to-one correspondence between terms in normal form and winning strategies.

2 Preliminaries

In this section we recall the definition of the (fragment of the) constructive modal logic CK we consider in this paper, and we recall the definition and some terminology for modal λ-terms. We are interested in a minimal constructive modal logic whose *formulas* are defined from a countable set of propositional variables $\mathcal{A} = \{a, b, c, \ldots\}$ using the following grammar:

$$A \coloneqq a \mid (A \to A) \mid \Box A \tag{4}$$

We say that a formula is *modality-free* if it contains no occurrences of the modality $\Box$. A formula is a $\to$*-formula* if it is of the form $A \to B$. In the following we use Krivine's convention [37] and write $(A_1, \ldots, A_n) \to C$ as a shortcut for $(A_1 \to (\cdots \to (A_n \to C)\cdots))$ A *sequent* is an expression $\Gamma \vdash C$ where Γ is a finite (possibly empty) list of formulas and C is a formula. If $\Gamma = A_1, \ldots, A_n$ and σ a permutation over $\{1, \ldots, n\}$, then we may write $\sigma(\Gamma)$ to denote $A_{\sigma(1)}, \ldots, A_{\sigma(n)}$.

In this paper we consider the logic CK defined by extending the conjunction-free and disjunction-free fragment of intuitionistic propositional logic with the modality $\Box$

$$\mathsf{ax}\,\frac{}{a \vdash a} \qquad \mathsf{ex}\,\frac{\Gamma \vdash C}{\sigma(\Gamma) \vdash C} \qquad \to^{\mathsf{R}}\,\frac{\Gamma, A \vdash C}{\Gamma \vdash A \to C} \qquad \to^{\mathsf{L}}\,\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, \Delta, A \to B \vdash C}$$

$$\mathsf{K}^{\Box}\,\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \qquad \mathsf{W}\,\frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \mathsf{C}\,\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \qquad \mathsf{cut}\,\frac{\Gamma \vdash A \quad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}$$

Fig. 2. Sequent calculus rules of the sequent system SCK, where σ is a permutation over $\{1, \ldots, n\}$

$$\mathsf{Id}\,\frac{}{x_1 : A_1, \ldots, x_n : A_n \vdash x_i : A_i}\, i \in \{1,\ldots,n\} \qquad \mathsf{Abs}\,\frac{\Gamma, x : A \vdash M : C}{\Gamma \vdash \lambda x.M : A \to C} \qquad \mathsf{App}\,\frac{\Gamma \vdash N : A \quad \Gamma \vdash M : A \to C}{\Gamma \vdash MN : C}$$

$$\Box\text{-}\mathsf{subst}\,\frac{\Gamma \vdash N_1 : \Box A_1 \quad \cdots \quad \Gamma \vdash N_n : \Box A_n \quad x_1 : A_1, \ldots, x_n : A_n \vdash M : C}{\Gamma \vdash M\,[N_1, \ldots, N_n / x_1, \ldots, x_n]_{\blacksquare} : \Box C}\, x_1,\ldots,x_n \text{ do not occur in } \Gamma$$

Fig. 3. Typing rules in the natural deduction system $\mathsf{ND}_{\mathsf{CK}}$ for modal λ-terms.

whose behavior is defined by the *necessitation rule* and the axiom K_1 below.

$$\mathsf{Nec} := \text{if } A \text{ is provable, then also } \Box A \text{ is} \qquad \mathsf{K}_1 := \Box(A \to B) \to (\Box A \to \Box B)$$

The sequent calculus SCK, whose rules are provided in Figure 2, is a sound and complete proof system for the logic CK. This system have been extracted from the one presented in [38] and satisfies cut-elimination.

2.1 A Lambda Calculus for CK

The set of (untyped) *modal λ-terms* is defined inductively from a countable set of *variables* $\mathcal{V} = \{x, y, \ldots\}$ using the following grammar:

$$M, N := x \mid \lambda x.M \mid (MN) \mid M\left[\vec{N}/\vec{x}\right]_{\blacksquare} \text{ where } \begin{cases} \vec{N} = N_1, \ldots, N_n \text{ is a list of terms and} \\ \vec{x} = x_1, \ldots x_n \text{ is a list of distinct variables.} \end{cases}$$

modulo the standard α-equivalence (denoted $=_\alpha$, see [8]) and modulo the equivalence generated by the following permutations (for any σ permutation over the set $\{1, \ldots, n\}$) over the order of substitutions in the $[\cdot/\cdot]_{\blacksquare}$ constructor:

$$\left[\vec{N}/\vec{x}\right]_{\blacksquare} := [N_1, \ldots, N_n/x_1, \cdots, x_n]_{\blacksquare} = [N_{\sigma(1)}, \ldots, N_{\sigma(n)}/x_{\sigma(1)}, \ldots, x_{\sigma(n)}]_{\blacksquare} =: \left[\sigma(\vec{N})/\sigma(\vec{x})\right]_{\blacksquare}$$
for any σ permutation over $\{1, \ldots, n\}$.

As usual, application associates to the left, and has higher precedence than abstraction. For example, $\lambda xyz.xyz := \lambda x.(\lambda y.(\lambda z.((xy)z)))$. A modal λ-term is a *(explicit) substitution* if it is of the form $M\left[\vec{N}/\vec{x}\right]_{\blacksquare}$, an *application* if of the form MN, and a *λ-abstraction* if of the form $\lambda x.M$.

The set of *subterms* of a term M (denoted $\mathsf{SUB}(M)$) is defined as follows:

$$\mathsf{Sub}(x) = \{x\} \quad , \quad \mathsf{Sub}(\lambda x.M) = \mathsf{Sub}(M) \cup \{\lambda x.M\} \quad , \quad \mathsf{Sub}(MN) = \mathsf{Sub}(M) \cup \mathsf{Sub}(N) \cup \{MN\}\,,$$
$$\mathsf{Sub}(M\,[N_1, \ldots, N_n/x_1, \ldots, x_n]_{\blacksquare}) = \mathsf{Sub}(M) \cup \left(\textstyle\bigcup_{i\in\{1,\ldots,n\}} \mathsf{Sub}(N_i)\right) \cup \{M\,[N_1, \ldots, N_n/x_1, \ldots, x_n]_{\blacksquare}\}\,.$$

Its *length* $|M|$ and its set of *free variables* $\mathtt{FV}(M)$ are defined as:

$$|M| = \begin{cases} 0 & \text{if } M = x \\ |N| + 1 & \text{if } M = \lambda x.N \\ \max\{|N|, |P|\} + 1 & \text{if } M = NP \\ \max\{|N|, |P_1|, \ldots, |P_n|\} + 1 & \text{if } M = N\left[\vec{P}/\vec{x}\right]_{\blacksquare} \end{cases} \qquad \mathtt{FV}(M) = \begin{cases} \{x\} & \text{if } M = x \\ \mathtt{FV}(N) \setminus \{x\} & \text{if } M = \lambda x.N \\ \mathtt{FV}(N) \cup \mathtt{FV}(P) & \text{if } M = NP \\ \bigcup_i \mathtt{FV}(P_i) & \text{if } M = N\left[\vec{P}/\vec{x}\right]_{\blacksquare} \end{cases}$$

We denote $|M|_x$ the number of the occurrences of the free variable x in a term M and we may write $|M|_x = 0$ if $x \notin \mathtt{FV}(M)$ and we say that a term M is *linear* in the variables $x_1, \ldots, x_n$ if $|M|_{x_i} = 1$ for all $i \in \{1, \ldots, n\}$. We denote by $M\{N_1, \ldots, N_n/x_1, \ldots, x_n\}$ the result of the standard capture avoiding substitution of the occurrences of the variable $x_1, \ldots, x_n$ in M with the term $N_1, \ldots, N_n$ respectively (see, e.g., [49]).

A *variable declaration* is an expression $x : A$ where x is a variable and A is a *type*, that is, a formula as defined in Equation (4). A *(typing) context* is a finite list $\Gamma \coloneqq x_1 : A_1, \ldots, x_n : A_n$ of distinct variable declarations. Given a context $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, we say that a variable x *appears* in Γ if $x = x_i$ for a $i \in \{1, \ldots, n\}$ and we denote by $\Gamma, y : B$ the context $x_1 : A_1, \ldots, x_n : A_n, y : B$ implicitly assuming that y does not appear in Γ. A *type assignment* is an expression of the form $\Gamma \vdash M : A$ where Γ is a context, M a modal λ-term and A a type.

Definition 1. *Let $\Gamma \vdash M : A$ be an type assignment. A* typing derivation *(or* derivation *for short) of $\Gamma \vdash M : A$ in* $\mathsf{ND_{CK}}$ *is a finite tree of type assignment constructed using the rules in Figure 3 in such a way it has root $\Gamma \vdash M : A$ and each leaf is the conclusion of a* Id*-rule. A type assignment is* derivable *(in* $\mathsf{ND_{CK}}$*) if there is a derivation with conclusion the given type assignment.*

We denote by Λ (resp. by $\Lambda^{\blacksquare}$ and Λ^{λ}) the set of modal λ-terms (resp. the set of substitutions and λ-abstractions in Λ) admitting a derivable type assignment in $\mathsf{ND_{CK}}$.

3 A New Modal Lambda Calculus

In this section we define a new modal lambda calculus by enriching the operational semantics of the previous calculi with additional reduction rules aiming at recovering canonicity, proving confluence and strong normalization properties.

To define our term rewriting rules, we require special care when they are applied in a proper sub-term. This is due to the fact that the explicit substitution encoded by $[\cdot/\cdot]_{\blacksquare}$ could capture free variables. For this reason, we introduce the notion of *term with a hole* as a term of the form $\mathbf{C}[\circ]$ containing a single occurrence of a special variable $\circ$. More precisely, the set CwH of terms with a hole and the two sets CwH_{η_1} and CwH_{η_2} of specific terms with a hole are defined by the following grammars:

$$\begin{array}{ll} \mathsf{CwH} & : \mathbf{C}[\circ] \coloneqq \circ \mid \lambda x.\mathbf{C}[\circ] \mid M\mathbf{C}[\circ] \mid \mathbf{C}[\circ]\,M \mid \mathbf{C}[\circ]\,[\vec{M}/\vec{x}] \mid M\left[\vec{N}_1, \mathbf{C}[\circ], \vec{N}_2/\vec{x}_1, x, \vec{x}_2\right]_{\blacksquare} \\ \mathsf{CwH}_{\eta_1} & : \mathbf{E}[\circ] \coloneqq \circ \mid \lambda x.\mathbf{E}[\circ] \mid M\mathbf{E}[\circ] \mid \mathbf{E}'[\circ]M \mid \mathbf{E}[\circ]\left[\vec{M}/\vec{x}\right]_{\blacksquare} \mid M\left[\vec{N}_1, \mathbf{E}, \vec{N}_2/\vec{x}_1, x, \vec{x}_2\right]_{\blacksquare} \\ \mathsf{CwH}_{\eta_2} & : \mathbf{D}[\circ] \coloneqq \circ \mid \lambda x.\mathbf{D}[\circ] \mid M\mathbf{D}[\circ] \mid \mathbf{D}[\circ]\,M \mid \mathbf{D}[\circ]\left[\vec{M}/\vec{x}\right]_{\blacksquare} \mid M\left[\vec{N}_1, \mathbf{D}'[\circ], \vec{N}_2/\vec{x}_1, x, \vec{x}_2\right]_{\blacksquare} \\ & \qquad\qquad \text{with } \mathbf{E}'[\circ] \neq [\circ] \neq \mathbf{D}'[\circ] \end{array}$$

We denote by $\mathbf{C}[M]$ the term obtained by replacing the hole $\circ$ in $\mathbf{C}[\circ]$ with the term M. By means of example, if $\mathbf{C}[\circ] = \circ$ then $\mathbf{C}[M] = M$ and if $\mathbf{E}[\circ] = (\lambda x.xN)[\circ/x]_{\blacksquare}$

Ground Steps:

$$(\lambda x.M)N \rightsquigarrow_{\beta_1} M\{N/x\}$$
$$M\left[\vec{P}, R\left[\vec{N}/\vec{z}\right]_\blacksquare, \vec{Q}/\vec{x}, y, \vec{w}\right]_\blacksquare \rightsquigarrow_{\beta_2} M\{R/y\}\left[\vec{P}, \vec{N}, \vec{Q}/\vec{x}, \vec{z}, \vec{w}\right]_\blacksquare$$
$$M \rightsquigarrow_{\eta_1} \lambda x.Mx \quad \text{if } \Gamma \vdash M : A \to B,\ x \notin \mathtt{FV}(M) \text{ and } M \notin \Lambda^\lambda$$
$$M \rightsquigarrow_{\eta_2} x\,[M/x]_\blacksquare \quad \text{if } \Gamma \vdash M : \Box A,\ x \notin \mathtt{FV}(M) \text{ and } M \notin \Lambda^\blacksquare$$
$$M\left[\vec{P}, N, \vec{Q}/\vec{x}, y, \vec{z}\right]_\blacksquare \rightsquigarrow_{\kappa_1} M\left[\vec{P}, \vec{Q}/\vec{x}, \vec{z}\right]_\blacksquare \quad \text{if } |M|_y = 0$$
$$M\left[\vec{P}, N, N, \vec{Q}/\vec{x}, y_1, y_2, \vec{z}\right]_\blacksquare \rightsquigarrow_{\kappa_2} M\{v, v/y_1, y_2\}\left[\vec{P}, N, \vec{Q}/\vec{x}, v, \vec{z}\right]_\blacksquare \quad \text{with } v \text{ fresh}$$

Reduction Steps in Contexts:

$$\frac{M \rightsquigarrow_{\beta_i} N}{\mathbf{C}[M] \rightsquigarrow_\beta \mathbf{C}[N]}\, {\scriptstyle i\in\{1,2\}} \qquad \frac{M \rightsquigarrow_{\kappa_i} N}{\mathbf{C}[M] \rightsquigarrow_\kappa \mathbf{C}[N]}\, {\scriptstyle i\in\{1,2\}} \qquad \frac{M \rightsquigarrow_{\eta_1} N}{\mathbf{E}[M] \rightsquigarrow_\eta \mathbf{E}[N]} \qquad \frac{M \rightsquigarrow_{\eta_2} N}{\mathbf{D}[M] \rightsquigarrow_\eta \mathbf{D}[N]}$$

with $\mathbf{C}[\circ] \in \mathsf{CwH}$ and $\mathbf{E}[\circ] \in \mathsf{CwH}_{\eta_1}$ and $\mathbf{D}[\circ] \in \mathsf{CwH}_{\eta_2}$

Fig. 4. Definition of the ground steps of the reduction relations, and the rules for their extension to terms with holes.

then $\mathbf{E}[M] = (\lambda x.xN)\,[M/x]_\blacksquare$. The reduction relations of our calculus are provided in Figure 4, where the ground steps and the rules for extending them to specific contexts are provided.

Remark 1. The term constructor Let (i.e., $[\cdot/\cdot]_\blacksquare$ from Equation (2)) plays no role in the standard η and β reduction rules from the literature, where it behaves as a black-box during reduction. The inertness of this constructor with respect to normalization is indeed what makes the lambda calculus in [9,31] unable to identify terms whose expected behavior is the same as, for example, the following pairs of terms:

$$x\,[v/x]_\blacksquare \quad \text{and} \quad x\,[v, w/x, y]_\blacksquare \qquad \Big| \qquad xyz\,[v, v/y, z]_\blacksquare \quad \text{and} \quad xyy\,[v/y]_\blacksquare \tag{5}$$

Our operational semantics extends the one provided in [31]. The novelty of our approach is the definition of the κ-reduction and the restriction of the η-reduction. The former is needed to being able to identify modal λ-terms with the same expected computational meaning, as the ones in Equation (5). The latter is carefully defined to avoid η-redexes that would make the reduction non-terminating, using a well-known technique in term rewriting theory (see, e.g., [42,30]).

The need of these restrictions can be observed in the two following (unrestricted) η-reduction chains, which are both forbidden by our restricted rule from Figure 4.

$$\underset{\text{whenever } \Gamma \vdash M : A \to B}{M \rightsquigarrow_\eta \lambda x.Mx \rightsquigarrow_\eta \lambda x.(\lambda y.My)x \rightsquigarrow_\eta \ldots} \quad \text{and} \quad \underset{\text{whenever } \Gamma \vdash M : \Box A}{M \rightsquigarrow_\eta x\,[M/x]_\blacksquare \rightsquigarrow_\eta x\,[y\,[M/y]_\blacksquare/x]_\blacksquare \rightsquigarrow_\eta \ldots}$$

Moreover, our definition rules out interactions between the η and β reductions which could lead to infinite chains, as the ones shown below.

$$\begin{array}{llll} \lambda x.M \rightsquigarrow_\eta \lambda y.(\lambda x.M)y \rightsquigarrow_\beta \lambda y.(M\{x/y\}) =_\alpha \lambda x.M & \text{or} \\ x\,[M/x]_\blacksquare \rightsquigarrow_\eta x\,[y\,[M/y]_\blacksquare/x]_\blacksquare \rightsquigarrow_\beta y\,[M/y]_\blacksquare =_\alpha x\,[M/x]_\blacksquare & . \end{array}$$

Definition 2. *We define the following reduction relations:*

$$\rightsquigarrow_{\beta\eta} = \rightsquigarrow_\beta \cup \rightsquigarrow_\eta \qquad \rightsquigarrow_{\beta\kappa} = \rightsquigarrow_\eta \cup \rightsquigarrow_\kappa \qquad \rightsquigarrow_{\beta\eta\kappa} = \rightsquigarrow_\beta \cup \rightsquigarrow_\eta \cup \rightsquigarrow_\kappa \tag{6}$$

For any $\xi \in \{\beta, \eta, \kappa, \beta\eta, \beta\kappa, \beta\eta\kappa\}$*, we denote by* $\rightsquigarrow_\xi^+$ *its* transitive closure*, by* $\rightsquigarrow_\xi^=$ *its* reflexive closure*, by* $\rightsquigarrow_\xi^*$ *its* reflexive and transitive closure*, and by* $\equiv_\xi$ *the* equivalence relation *it enforces over terms, that is, its reflexive, symmetric and transitive closure. Given a term M, we denote by* $\mathsf{nf}_\xi(M)$ *the set of its* $\rightsquigarrow_\xi$-normal form*. A term M is* strongly normalizable *for* $\rightsquigarrow_\xi$ *if it admits no infinite* $\rightsquigarrow_\xi$*-chains A reduction* $\rightsquigarrow_\xi$ *is* strongly normalizing *if every term M is strongly normalizable for it. A reduction* $\rightsquigarrow_\xi$ *is* confluent *if given* $M \rightsquigarrow_\xi^* N_1$ *and* $M \rightsquigarrow_\xi^* N_2$ *there exists a term N such that* $N_1 \rightsquigarrow_\xi^* N$ *and* $N_2 \rightsquigarrow_\xi^* N$*.*

The *substitution* lemma and *subject reduction* theorem holds for the reduction $\rightsquigarrow_{\beta\eta\kappa}$.

Lemma 1. *[Substitution Lemma] Let* $\Gamma, x : B \vdash M : C$ *and* $\Gamma \vdash N : B$ *be derivable type assignments. Then* $\Gamma, x : B \vdash M\{N/x\} : C$ *is a derivable type assignment.*

Proof. The proof is by induction on $|M|$.

- If $|M| = 1$ is a variable z we either have that $M = x$ or $M = y$ for some $y \neq x$ that appears in Γ. Then either $M\{N/x\} = N$ and $C = B$, or $M\{N/x\} = M$. In both cases $\Gamma \vdash M\{N/x\} : C$ is derivable by hypothesis.
- If $|M| \geq 1$ and M is an abstraction or an application, then the proof is the same as in standard λ-calculus [49]. If $M = P[T_1, \ldots, T_n/x_1, \ldots x_n]_\blacksquare$ then $C = \Box C'$ and $M\{N/x\} = P[T_1\{N/x\}, \ldots T_n\{N/x\}/x_1, \ldots x_n]_\blacksquare$. Then, by definition of derivability, we have $x \notin \{x_1, \ldots, x_n\}$ and the type assignments $x_1 : A_1, \ldots, x_n : A_n \vdash M : C$ and $\Gamma, x : B \vdash N_i : \Box A_i$ are derivable for all $i \in \{1, \ldots, n\}$ and for some $A_1, \ldots, A_n$. We can apply inductive hypothesis on $\Gamma, x : B \vdash N_i : \Box A_i$ to deduce that the type assignment $\Gamma \vdash T_i\{N/x\} : \Box A_i$ is derivable for all $i \in \{1, \ldots, n\}$. We conclude the existence the desired type assignment with bottom-mots rule a $\Box$-subst with premises $x_1 : A_1, \ldots, x_n : A_n \vdash \{M/x\} : C$ and $\Gamma \vdash T_i\{N/x\} : \Box A_i$ for all $i \in \{1, \ldots, n\}$.

Theorem 1. *Let* $\Gamma \vdash M : C$ *be derivable. If* $M \rightsquigarrow_{\beta\eta\kappa} N$*, then* $\Gamma \vdash N : C$*.*

Proof. Because of Lemma 1, it suffices to check the cases when M reduces to N in one ground step of $\rightsquigarrow_{\beta\eta\kappa}$:

- if $M \rightsquigarrow_{\beta_1} N$, then $M = (\lambda x.P)Q$ and $N = P\{Q/x\}$. The case where $M \rightsquigarrow_{\beta_2} N$ uses a similar argument. The result follows the fact that if $\Gamma, x : B \vdash M : C$ and $\Gamma \vdash N : B$ are derivable type assignment, then $\Gamma, x : B \vdash M\{N/x\} : C$ by Lemma 1.
- if $M \rightsquigarrow_{\eta_1} N$, then $C = A \to B$ and $N = \lambda x.Mx$. The result follows by applying the rule Abs. The case where $M \rightsquigarrow_{\eta_2} N$ uses a similar argument;
- if $M \rightsquigarrow_{\kappa_1} N_1$, then $M = M'[P_1, \ldots, P_k, N, P_{k+1}, \ldots, P_n/x_1, \ldots, x_k, x, x_{k+1}, \ldots, x_n]_\blacksquare$ such that x is not free in M, $C = \Box B$, and $N_1 = M'\left[\vec{P}, \vec{Q}/\vec{x}, \vec{y}\right]_\blacksquare$. Then there are derivations for $\Gamma \vdash P_i : A_i$ for all $i \in \{1, \ldots, n\}$ (for some A_i) and a derivation for $x_1 : A_1, \ldots, x_k : A_k, x : A, x_{k+1} : A_{k+1} \ldots, x_n : A_n \vdash M' : B$. Therefore we have a derivation for $x_1 : A_1, \ldots, x_n : A_n \vdash M' : B$ since weakening is admissible (that is, whenever $\Gamma, x : A \vdash M : C$ is derivable and x does not occur free in M, then

$\Gamma \vdash M : C$ is also derivable[5]. Then we have a derivation of $\Gamma \vdash N : C$ with bottom-most rule a $\Box$-subst with right-most premise $x_1 : A_1, \dots, x_n : A_n \vdash M' : B$. and a premise $\Gamma \vdash P_i : A_i$ for each $i \in \{1, \dots, n\}$;

- if $M \rightsquigarrow_{\kappa_2} N_1$, then we conclude similarly to the previous point since we have

$$M = M' \left[\vec{P}, N, N, \vec{Q}/\vec{x}, y_1, y_2, \vec{z}\right]_{\blacksquare} \quad \text{and} \quad N_1 = M\{y, y/y_1, y_2\}\left[\vec{P}, N, \vec{Q}/\vec{x}, y, \vec{z}\right]_{\blacksquare} .$$

We can prove local confluence of $\rightsquigarrow_{\beta\eta\kappa}$ by case analysis of the critical pairs using the following lemma.

Lemma 2. *Let P, P' and Q modal λ-terms. If $P \rightsquigarrow_{\beta\eta\kappa} P'$, then $P\{Q/x\} \rightsquigarrow^*_{\beta\eta\kappa} P'\{Q/x\}$. Moreover, there is a N_Q such that $Q\{P/x\} \rightsquigarrow^*_{\beta\eta\kappa} N_Q$ and $Q\{P'/x\} \rightsquigarrow^*_{\beta\eta\kappa} N_Q$.*

Proof. To prove that $P\{Q/x\} \rightsquigarrow^*_{\beta\eta\kappa} P'\{Q/x\}$ we first prove that the result holds for the ground reduction steps:

- If $P = (\lambda y.M)N \rightsquigarrow_{\beta_1} M\{N/y\} = P'$, then we have $P\{Q/x\} = ((\lambda y.M)N)\{Q/x\} = (\lambda y.M\{Q/x\})(N\{Q/x\})$. We conclude by associativity of substitution.
- If $P \rightsquigarrow_{\beta_1} P'$, we conclude similarly to the previous case.
- If $P \rightsquigarrow_{\eta_1} \lambda y.Py = P'$, then $(\lambda y.Py)\{Q/x\} = \lambda y.(P\{Q/x\})y$. We conclude since, definition of $\rightsquigarrow_{\eta}$, we have that $P\{Q/x\}\{q/x\} \rightsquigarrow_{\eta} \lambda y.(P\{Q/x\})y$.
- If $P \rightsquigarrow_{\eta_2} y[P/y]_{\blacksquare} = P'$, then $P'\{Q/x\} = y[(P\{Q/x\})/y]_{\blacksquare}$. We conclude since, by definition of $\rightsquigarrow_{\eta}$, we have that $P\{Q/x\} \rightsquigarrow_{\eta} y[(P\{Q/x\})/y]_{\blacksquare}$.
- If $P = M'\left[\vec{P}, N, \vec{Q}/\vec{x}, y, \vec{z}\right]_{\blacksquare} \rightsquigarrow_{\kappa_1} P' = M'\left[\vec{P}, \vec{Q}/\vec{x}, \vec{z}\right]_{\blacksquare}$, then $P\{Q/x\} = M'\{Q/x\}\left[\vec{P}\{Q/x\}, N\{Q/x\}, \vec{Q}\{Q/x\}/\vec{x}, y, \vec{z}\right]_{\blacksquare}$. We conclude since $P'\{Q/x\} = M'\{Q/x\}\left[\vec{P}\{Q/x\}, \vec{Q}\{Q/x\}/\vec{x}, \vec{z}\right]_{\blacksquare}$, thus $P\{Q/x\} \rightsquigarrow_{\kappa_1} P'\{Q/x\}$.
- If $P = M'\left[\vec{P}, N, N, \vec{Q}/\vec{x}, y_1, y_2, \vec{z}\right]_{\blacksquare} \rightsquigarrow_{\kappa_2} P' = M'\{v, v/y_1, y_2\}\left[\vec{P}, N, \vec{Q}/\vec{x}, v, \vec{z}\right]_{\blacksquare}$, then $P\{Q/x\} = M'\{Q/x\}\left[\vec{P}\{Q/x\}, N\{Q/x\}, N\{Q/x\}, \vec{Q}\{Q/x\}/\vec{x}, y_1, y_2, \vec{z}\right]_{\blacksquare}$ and $P'\{Q/x\} = (M'\{v, v/y_1, y_2\})\{Q/x\}\left[\vec{P}\{Q/x\}, N\{Q/x\}, \vec{Q}\{Q/x\}/\vec{x}, v, \vec{z}\right]_{\blacksquare}$. We conclude since $(M'\{v, v/y_1, y_2\})\{Q/x\} = (M'\{Q/x\})\{v, v/y_1, y_2\}$, thus $P\{Q/x\} \rightsquigarrow_{\kappa_2} P'\{Q/x\}$.

Then we conclude by showing that it also holds when reductions are applied in a context.

- If $P = \lambda y.M$ for a M such that $M \rightsquigarrow_{\beta\eta\kappa} M'$, then $P' = \lambda y.M'$ and we conclude by inductive hypothesis since $P\{Q/x\} = \lambda y.M\{Q/x\} \rightsquigarrow_{\beta\eta\kappa} \lambda y.M'\{Q/x\}$.
- If $P = MN$, then $P\{Q/x\} = M\{Q/x\}N\{Q/x\}$. In this case, either $M \rightsquigarrow_{\beta\eta\kappa} M'$ and $P' = M'N$, or $N \rightsquigarrow_{\beta\eta\kappa} N'$ and $P' = MN'$. We conclude taking into account the restriction of the possible application of the reduction steps in a context. Note that without the restriction on $\rightsquigarrow_{\eta_1}$ we could have had $M\{Q/x\} \rightsquigarrow_{\eta_1} \lambda y.(M\{Q/x\})y$, and therefore $M\{Q/x\}N \rightsquigarrow_{\eta_1} \lambda y.(M\{Q/x\})yN \rightsquigarrow_{\beta_1} M\{Q/x\}N$;

[5] The admissibility of weakening is easily proven by induction on the size of a derivation.

- If $P = M[N/y]_\blacksquare$ then $P\{Q/x\} = M\{Q/x\}[N\{Q/x\}/y]_\blacksquare$. In this case, either $M \rightsquigarrow_{\beta\eta\kappa} M'$ and $P' = M'[N/y]_\blacksquare$, or $N \rightsquigarrow_{\beta\eta\kappa} N'$ and $P' = M[N'/y]_\blacksquare$. We conclude taking into account the restriction of the possible application of the reduction steps in a context. Note that without the restriction on $\rightsquigarrow_{\eta_2}$ we could have had $N\{Q/x\} \rightsquigarrow_{\eta_2} y[N\{Q/x\}/y]_\blacksquare$, and therefore the following sequence of reductions.

$$M[N\{Q/x\}/z]_\blacksquare \rightsquigarrow_{\eta_2} M[y[N\{Q/x\}/y]_\blacksquare/z]_\blacksquare \rightsquigarrow_{\beta_2} M[N\{Q/x\}/z]_\blacksquare$$

The fact that, for each Q, there is a N_Q such that $Q\{P/x\} \rightsquigarrow^*_{\beta\eta\kappa} N_Q$ and $Q\{P'/x\} \rightsquigarrow^*_{\beta\eta\kappa} N_Q$ is proven by induction on the structure of Q and considering the restrictions on the definition of the rewriting steps in a context.

Proposition 1. *The reduction $\rightsquigarrow_{\beta\eta\kappa}$ is locally confluent.*

Proof. We show that if there are M, N_1 and N_2 with $N_1 \neq N_2$ such that $M \rightsquigarrow_{\beta\eta\kappa} N_1$ and $M \rightsquigarrow_{\beta\eta\kappa} N_2$, then there exists N such that $N_1 \rightsquigarrow^*_{\beta\eta\kappa} N$ and $\rightsquigarrow^*_{\beta\eta\kappa} N$. Without loss of generality we have the following cases:

1. if $M \rightsquigarrow_{\beta_1} N_1$ with $M = (\lambda x.P)Q$ and $N_1 = P\{Q/x\}$, then N_2 can only be obtained by applying $\rightsquigarrow_{\beta\eta\kappa}$ the subterms P and Q of M. We conclude by Lemma 2;
2. if $M \rightsquigarrow_{\beta_2} N_1$ with $M = M'\left[\vec{P}, R\left[\vec{N}/\vec{z}\right]_\blacksquare, \vec{Q}/\vec{x}, y, \vec{w}\right]_\blacksquare$ and with $N_1 = M'\{R/y\}\left[\vec{P}, \vec{N}, \vec{Q}/\vec{x}, \vec{z}, \vec{w}\right]_\blacksquare$, then N_2 must be a term obtained by applying $\rightsquigarrow_{\beta\eta\kappa}$ on R or on one of the terms in $\vec{P}$, $\vec{N}$ or $\vec{Q}$. We conclude again by Lemma 2;
3. if $M \rightsquigarrow_{\eta_1} N_1$, then $\Gamma \vdash M : A \to B$ and $N_1 = \lambda x.Mx$. Therefore, for any N_2 such that $M \rightsquigarrow_{\beta\eta\kappa} N_2$ we have that $\Gamma \vdash N_2 : A \to B$ (by subject reduction). Then
   - either N_2 is not an abstraction and we conclude by letting $N = \lambda x.N_2x$.
   - otherwise $N_2 = \lambda y.M'$ and we conclude since $N_1 \rightsquigarrow_{\eta_1} \lambda x.N_2x \rightsquigarrow_{\beta_1} N_2$.
4. if $M \rightsquigarrow_{\eta_2} N_1$ with $\Gamma \vdash M : \Box A$ and $N_1 = x[M/x]_\blacksquare$, then we conclude with a similar argument with respect to the previous point by letting $N = x[N_2/x]_\blacksquare$.
5. if $M \rightsquigarrow_\kappa N_1$, $M = M'\left[\vec{P}, N, \vec{Q}/\vec{x}, x, \vec{y}\right]_\blacksquare \rightsquigarrow_{\kappa_1} M'\left[\vec{P}, \vec{Q}/\vec{x}, \vec{y}\right]_\blacksquare = N_1$, or $M = M'\left[\vec{P}, N, N, \vec{Q}/\vec{x}, y_1, y_2, \vec{z}\right]_\blacksquare \rightsquigarrow_{\kappa_2} M\{y, y/y_1, y_2\}\left[\vec{P}, N, \vec{Q}/\vec{x}, y, \vec{z}\right]_\blacksquare = N_1$. In both cases we conclude with an argument similar to the one in Case (2).

In order to prove the termination of $\rightsquigarrow_{\beta\eta\kappa}$, we define the following measures.

Definition 3. *Let M be a modal λ-term. We define the following multisets of derivable type assignments:*

$$\mathsf{Est}_1(M) = \left\{B \to C \mid P \in \mathsf{Sub}(M) \setminus \Lambda^\lambda \text{ such that } M \neq PQ \text{ and } \Gamma \vdash P : B \to C\right\}$$
$$\mathsf{Est}_2(M) = \left\{\Box B \mid P \in \mathsf{Sub}(M) \setminus \Lambda^\blacksquare \text{ such that } M \neq Q\left[\vec{N}_1, P, \vec{N}_2/\vec{x}_1, x, \vec{x}_2\right]_\blacksquare \text{ and } \Gamma \vdash P : \Box B\right\}$$

We then define $\|M\|_\eta \coloneqq \|M\|^1_\eta + \|M\|^2_\eta$ with

$$\|M\|^1_\eta \coloneqq \sum_{A \in \mathsf{Est}_1(M)} \|A\|^1_\eta \quad \text{and} \quad \|M\|^2_\eta \coloneqq \sum_{A \in \mathsf{Est}_2(M)} \|A\|^2_\eta$$

where

$$\begin{array}{lll} \|a\|^1_\eta = 0 & \|A \to B\|^1_\eta = \|A\|^1_\eta + \|B\|^1_\eta + 1 & \|\Box A\|^1_\eta = \|A\|^1_\eta \\ \|a\|^2_\eta = 0 & \|A \to B\|^2_\eta = \|A\|^2_\eta + \|B\|^2_\eta & \|\Box A\|^2_\eta = \|A\|^2_\eta + 1 \end{array}$$

We also define $\|M\|_\kappa$ as the size of substitution subterms of M as follows:

$$\|x\|_\kappa = 0 \qquad \|\lambda x M\|_\kappa = \|M\|_\kappa \qquad \|MN\|_\kappa = \|M\|_\kappa + \|N\|_\kappa$$
$$\|M\,[N_1,\ldots,N_n/x_1,\ldots,x_n]_\blacksquare\,\|_\kappa = \|M\|_\kappa + \|N\|_\kappa + n$$

Example 1. Intuitively, the measure $\|\cdot\|_\eta$ does not take into account all the subterms of M, but only the ones on which we can apply the restricted $\rightsquigarrow_\eta$. For an example, consider the modal λ-term $M = (\lambda z^{a\to a}.z)y$ with $\|M\|_\eta = 3$ because all four subterms of M are of type a $\to$-formula, but the subterm $\lambda z.z$ is an abstraction, therefore no $\rightsquigarrow_\eta$ can be applied on it. If $M \rightsquigarrow_\eta N$, because of the restrictions on $\rightsquigarrow_\eta$, we have that

- either $N = (\lambda z.z)(\lambda v.yv)$ with $\|N\|_\eta = 2$ because no $\rightsquigarrow_\eta$ can be applied to the subterms y and $\lambda z.z$ (they occur on the left of an application) or $\lambda v.yv$ (it is an abstraction), but only to the subterms z and the whole term N;
- or $N = \lambda v^a.((\lambda z.z)y)v$ with $\|N\|_\eta = 2$ because $\rightsquigarrow_\eta$ can only be applied to the subterms z and y.

Lemma 3. *Let M and N be modal λ-terms. If $M \rightsquigarrow_\eta N$, either $\|N\|_\eta < \|M\|_\eta$ or there is N' such that $N \rightsquigarrow_\eta N'$ and $\|N'\|_\eta < \|M\|_\eta$.*

Proof. We only discuss the two following cases, since the others are direct consequence of the definitions of $\|\cdot\|^1_\eta$ and $\|\cdot\|^2_\eta$.

1. If $\Gamma \vdash M : C$ with $C = A \to B$ and $M \rightsquigarrow_{\eta_1} N$, then $N = \lambda x.Mx$. Therefore, $\|C\|^1_\eta = \|A \to B\|^1_\eta > 0$ and

$$\|M\|^1_\eta = \left(\sum_{C'\in\mathsf{Est}_1(M)} \|C'\|^1_\eta\right) > \sum_{C'\in\mathsf{Est}_1(M)\setminus\{C\}} \|C'\|^1_\eta > \|\lambda x.Mx\|^1_\eta = \|N\|^1_\eta$$

   Now consider $\|N\|^2_\eta$. We reason by cases on the structure of the type A. If A is not a box, we can conclude by setting $N' = N$. If A is of the shape $\Box A'$ then $\|N\|^2_\eta = \sum_{C'\in\mathsf{Est}_2(M)\cup\{A\}} \|C'\|^2_\eta$. This means that we can perform a step $\lambda x.Mx \rightsquigarrow_{\eta_2} \lambda x.(M(z\,[x/z]_\blacksquare))$ with z fresh. Now we set $N' = \lambda x.(M(z\,[x/z]_\blacksquare))$ since $\|N'\|^2_\eta = \sum_{C'\in\mathsf{Est}_2(M)} \|C'\|^2_\eta = \|M\|^2_\eta$. We remark that $\|N'\|^1_\eta = \|N\|^1_\eta$. Hence $\|N'\|_\eta = \|N\|^1_\eta + \|M\|^2_\eta < \|M\|^1_\eta + \|M\|^2_\eta = \|M\|_\eta$ since $\|N\|^1_\eta < \|M\|^1_\eta$ for what we said before.
2. If $\Gamma \vdash M : C$ with $C = \Box A$ and $M \rightsquigarrow_{\eta_2} N$, then $N = x\,[M/x]_\blacksquare$. Therefore, $\|C\|^2_\eta = \|\Box A\|^2_\eta > 0$ and

$$\|M\|^2_\eta = \left(\sum_{C'\in\mathsf{Est}_1(M)} \|C'\|^2_\eta\right) > \sum_{C'\in\mathsf{Est}_1(M)\setminus\{C\}} \|C'\|^1_\eta > \|\lambda x.Mx\|^2_\eta = \|x\,[M/x]_\blacksquare\,\|^2_\eta = \|N\|^2_\eta$$

   Now consider $\|N\|^1_\eta$. We reason by cases on the structure of the type A. If A is not an implication, we can conclude by setting $N' = N$. If A is of the shape $A' \to B$ then $\|N\|^1_\eta = \sum_{C'\in\mathsf{Est}_1(M)\cup\{A\}} \|C'\|^2_\eta$. This means that we can perform a step $x\,[M/x]_\blacksquare \rightsquigarrow_{\eta_2} (\lambda y.xy)\,[M/x]_\blacksquare$ with y fresh. Now we set $N' = (\lambda y.xy)\,[M/x]_\blacksquare$ since $\|N'\|^1_\eta = \sum_{C'\in\mathsf{Est}_1(M)} \|C'\|^2_\eta = \|M\|^1_\eta$. We remark that $\|N'\|^2_\eta = \|N\|^2_\eta$. Hence $\|N'\|_\eta = \|N\|^2_\eta + \|M\|^1_\eta < \|M\|^2_\eta + \|M\|^1_\eta = \|M\|_\eta$ since $\|N\|^2_\eta < \|M\|^2_\eta$ for what we said before.

Lemma 4. *Let P, Q, N be modal λ-terms. If $P\{Q/x\} \rightsquigarrow_\eta N$, then there are N_1 and N_2 such that $P \rightsquigarrow_\eta^* N_1$ and $Q \rightsquigarrow_\eta^* N_2$ with $N_1\{N_2/x\} = N$.*

Proof. We prove it by induction on the structure of P.

- If $P = x$ then $P\{Q/x\} = Q \rightsquigarrow_\eta N$. Then $N_1 = x$ and $N_2 = N$.
- If $P = \lambda y.P'$, then $P\{Q/x\} = \lambda y.P'\{Q/x\}$. By definition, if $P\{Q/x\} \rightsquigarrow_\eta N$ then $\Gamma \vdash P : A \to B$ and then $N = \lambda z.P''$ with $P'\{Q/x\} \rightsquigarrow_\eta P''$. In this case we apply inductive hypothesis to get N_1' and N_2' such that $P' \rightsquigarrow_\eta N_1'$ and $Q \rightsquigarrow_\eta N_2'$ with $N_1'\left\{N_2'/x\right\} = P''$. We conclude by letting $N_1 = \lambda y.N_1'$ and $N_2 = N_2'$;
- If $P = ST$, then $P\{Q/x\} = S\{Q/x\}T\{Q/x\}$. By definition of $\rightsquigarrow_\eta$, if $P\{Q/x\} \rightsquigarrow_\eta N$ then:
  - if $S\{Q/x\}T\{Q/x\} \rightsquigarrow_{\eta_1} \lambda y.(S\{Q/x\}T\{Q/x\})y = N$, then $N_1 = \lambda y(ST)y$ and $N_2 = Q$;
  - if $S\{Q/x\}T\{Q/x\} \rightsquigarrow_{\eta_2} y\left[S\{Q/x\}T\{Q/x\}/y\right]_\blacksquare = N$, then $N_1 = y\left[ST/y\right]_\blacksquare$ and $N_2 = Q$;
  - otherwise, the step $N \rightsquigarrow_\eta N'$ must be applied in a context. In this case, we have that $N = P_1P_2$ and either $S\{Q/x\} \rightsquigarrow_\eta P_1$ or $T\{Q/x\} \rightsquigarrow_\eta P_2$. In the case $S\{Q/x\} \rightsquigarrow_\eta P_1$, by definition of the contextual step, $\{Q/x\} \rightsquigarrow_\eta P_1$ cannot be a step of $\rightsquigarrow_{\eta_1}$. By inductive hypothesis there are $N_{1,2}$ and $N_{2,2}$ such that $S \rightsquigarrow_\eta N_{1,2}$ and $Q \rightsquigarrow_\eta N_{2,2}$ with $P_1 = N_{1,1}\{N_{2,2}/x\}$. Then $N_1 = N_{1,1}(T\{Q/x\})$ and $N_2 = N_{2,2}$. The other case is similar.
- If $P = S\left[\vec{T}/\vec{y}\right]_\blacksquare$ then $P\{Q/x\} = S\{Q/x\}\left[\vec{T}\{Q/x\}/\vec{y}\right]_\blacksquare$. By definition of $\rightsquigarrow_\eta$, if $P\{Q/x\} \rightsquigarrow_\eta N$ then:
  - if $S\{Q/x\}\left[\vec{T}\{Q/x\}/\vec{y}\right]_\blacksquare \rightsquigarrow_{\eta_1} \lambda y.(S\{Q/x\}\left[\vec{T}\{Q/x\}/\vec{y}\right]_\blacksquare)y = N$, then $N_1 = \lambda y.(S\left[\vec{T}/\vec{y}\right]_\blacksquare)y$ and $N_2 = Q$;
  - otherwise, the step $N \rightsquigarrow_\eta N'$ must be applied in a context. In this case, we have that $N = P_1\left[\vec{P}_2/\vec{y}\right]_\blacksquare$ and either $S\{Q/x\} \rightsquigarrow_\eta P_1$ or $\vec{T}\{Q/x\} \rightsquigarrow_\eta \vec{P}_2$. We do the case $S\{Q/x\} \rightsquigarrow_\eta P_1$. By inductive hypothesis there exists $N_{1,2}$ and $N_{2,2}$ such that $S \rightsquigarrow_\eta N_{1,2}$ and $Q \rightsquigarrow_\eta N_{2,2}$ with $P_1 = N_{1,1}\{N_{2,2}/x\}$. Then $N_1 = N_{1,1}\left[\vec{T}\{Q/x\}/\vec{y}\right]_\blacksquare$ and $N_2 = N_{2,2}$. The other case is similar.

Lemma 5. *The following commutations between $\rightsquigarrow_\beta$, $\rightsquigarrow_\eta$ and $\rightsquigarrow_\kappa$ hold:*

- *if $M \rightsquigarrow_\kappa N \rightsquigarrow_\beta N'$, then there is M' such that $M \rightsquigarrow_\beta M'$ and $M' \rightsquigarrow_\kappa^* N'$;*
- *if $M \rightsquigarrow_\eta N \rightsquigarrow_\kappa N'$, then there is M' such that $M \rightsquigarrow_\kappa M'$ and $M' \rightsquigarrow_\eta^* N'$;*
- *if $M \rightsquigarrow_\beta N \rightsquigarrow_\eta N'$, then there is M' such that $M \rightsquigarrow_\eta M'$ and $M' \rightsquigarrow_\beta^* N'$.*

Proof. After Lemmas 1 and 4, we can reason on the structure of M only considering its possible shape according to the ground steps of $\rightsquigarrow_{\beta\eta\kappa}$.

- If $M \rightsquigarrow_{\eta_1} N = \lambda x.Mx$, then, by lemma 4 we have N_1 and N_2 such that $P \rightsquigarrow_\eta N_1$ and $Q \rightsquigarrow_\eta N_2$ with $N_1\{N_2/x\} = N$. We conclude by letting $M' = (\lambda x.N_1)N_2$. A similar argument is applied if $M \rightsquigarrow_{\eta_2} N$.
- If $M = (\lambda x.P)Q \rightsquigarrow_{\beta_1} P\{Q/x\} = N$, then we have N_1 and N_2 such that $P \rightsquigarrow_\eta N_1$ and $Q \rightsquigarrow_\eta N_2$ with $N_1\{N_2/x\} = N$. We conclude by letting $M' = (\lambda x.N_1)N_2$. A similar argument is applied if $M \rightsquigarrow_{\beta_2} N$.

– If $M = M\left[\vec{P}, N, \vec{Q}/\vec{x}, y, \vec{y}\right]_{\blacksquare} \rightsquigarrow_{\kappa_1} M\left[\vec{P}, \vec{Q}/\vec{x}, \vec{y}\right]_{\blacksquare} = N$, then we can conclude since any β-redex of N is also a β-redex of M. A similar argument is applied if $M \rightsquigarrow_{\kappa_2} N$.

Therefore, the following corollary trivially holds.

Corollary 1. *The following hold.*

– *if $M \rightsquigarrow_{\kappa} N$, then $\mathsf{nf}_{\beta}(M) \rightsquigarrow^{*}_{\kappa} \mathsf{nf}_{\beta}(N)$;*
– *if $M \rightsquigarrow_{\eta} N$, then $\mathsf{nf}_{\kappa}(M) \rightsquigarrow_{\eta} \mathsf{nf}_{\kappa}(N)$;*
– *if $M \rightsquigarrow_{\beta} N$, then $\mathsf{nf}_{\eta}(M) \rightsquigarrow^{*}_{\beta} \mathsf{nf}_{\eta}(N)$.*

Theorem 2. *The reduction relation $\rightsquigarrow_{\beta\eta\kappa}$ is strongly normalizing and confluent.*

Proof. After Proposition 1, it suffices to prove that $\rightsquigarrow_{\beta\eta\kappa}$ is strongly normalizing to conclude by Newman's lemma that $\rightsquigarrow_{\beta\eta\kappa}$ is also confluent.

To prove strong normalization we use the fact that the reductions $\rightsquigarrow_{\beta}$, $\rightsquigarrow_{\eta}$ and $\rightsquigarrow_{\kappa}$ are strongly normalizing: for $\rightsquigarrow_{\beta}$ the proof can be found in [31], for $\rightsquigarrow_{\eta}$ the proof is by induction on $\|\cdot\|_{\eta}$ using Lemma 3, and for $\rightsquigarrow_{\kappa}$ it follows the fact that, by definition of $\|\cdot\|_{\kappa}$, we have that $\|M\|_{\kappa} > \|N\|_{\kappa}$ whenever $M \rightsquigarrow_{\kappa} N$. To conclude that $\rightsquigarrow_{\beta\eta\kappa}$ also is strongly normalizing, the standard result (see, e.g., [49]) in rewriting theory ensuring that given two strongly normalizing reduction relations $\rightsquigarrow_1$ and $\rightsquigarrow_2$ with $\rightsquigarrow_1$ confluent, if $M \rightsquigarrow_2 N$ implies the existence of a reduction $\mathsf{nf}_1(M) \rightsquigarrow^{+}_{2} \mathsf{nf}_1(N)$ for any M and N, , then $\rightsquigarrow_1 \cup \rightsquigarrow_2$ is strongly normalizing. In our case, the fact that $M \rightsquigarrow_2 N$ implies $\mathsf{nf}_1(M) \rightsquigarrow^{+}_{2} \mathsf{nf}_1(N)$ is a corollary of Lemma 5.

Definition 4. *The set $\widehat{\Lambda}$ is the set of modal λ-terms defined inductively as follows:*

– *if x is a variable, $T_1, \ldots, T_n \in \widehat{\Lambda}$, and there are derivations for the types assignments $\Gamma \vdash x : (A_1, \ldots, A_n) \to C$ with C atomic and $\Gamma \vdash T_i : A_i$ for all $i \in \{1, \ldots, n\}$, then $xT_1 \cdots T_n \in \widehat{\Lambda}$. Variables are the special case with $n = 0$;*
– *if $T \in \widehat{\Lambda}$ and there is a derivation of $\Gamma, x : A \vdash T : C$, then $\lambda x^{A}.T \in \widehat{\Lambda}$;*
– *if $M \in \widehat{\Lambda}$, $\mathtt{FV}(M) = \{x_1, \ldots, x_n\}$ and the type assignment $x_1 : B_1, \ldots, x_n : B_n \vdash M : C$ is derivable, and if there are n distinct terms $T_1, \ldots, T_n \in \Lambda$ of the shape $T_i = y_i U_{i1} \cdots U_{ik_i}$ with $U_{ij} \in \widehat{\Lambda}$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k_i\}$, such that the type assignment $\Gamma \vdash T_i : \Box B_i$ is derivable for all $i \in \{1, \ldots, n\}$, then $M[T_1, \ldots, T_n/x_1, \ldots, x_n]_{\blacksquare} \in \widehat{\Lambda}$.*

Proposition 2. *The set $\widehat{\Lambda}$ is the set of modal λ-terms in $\beta\eta\kappa$-normal form $\mathsf{nf}_{\beta\eta\kappa}(\Lambda)$.*

Proof. By definition, every $\widehat{\Lambda} \subseteq \mathsf{nf}_{\beta\eta\kappa}(\Lambda)$ is $\rightsquigarrow_{\beta\eta\kappa}$-normal. To prove the converse we proceed by induction on the structure of $M \in \mathsf{nf}_{\beta\eta\kappa}(\Lambda)$:

– if $M = x$, then $M \in \widehat{\Lambda}$ by definition;
– if $M = \lambda x.M' \in \mathsf{nf}_{\beta\eta\kappa}(\Lambda)$, then also $M' \in \mathsf{nf}_{\beta\eta\kappa}(\Lambda)$. By inductive hypothesis, this implies $M' \in \widehat{\Lambda}$. Therefore $\lambda x.M' \in \widehat{\Lambda}$;

$$\mathsf{ax}\,\frac{}{\Gamma, x : c \vdash x : c} \qquad \mathsf{ex}\,\frac{\Gamma \vdash M : C}{\sigma(\Gamma) \vdash M : C}\,* \qquad \mathsf{K}^{\Box}\,\frac{x_1 : A_1, \ldots, x_n : A_n \vdash M : C}{\Delta, y_1 : \Box A_1, \ldots, y_n : \Box A_n \vdash M\,[x_1, \ldots, x_n / y_1, \ldots, y_n]_{\blacksquare} : \Box C}\,\star$$

$$\to_{\mathsf{L}}^{\mathsf{ax}}\,\frac{\{\Gamma, y : B \vdash N_i : A_i\}_{i \in \{1,\ldots,n\}}}{\Gamma, y : \underbrace{(A_1, \ldots, A_n) \to c}_{B} \vdash y N_1 \cdots N_n : c}\,\S \qquad \to_{\mathsf{R}}^{*}\,\frac{\Gamma, x_1 : A_1, \ldots, x_n : A_n \vdash M : C}{\Gamma \vdash \lambda x_1^{A_1} \cdots x_n^{A_n}.M : (A_1, \ldots A_n) \to C}$$

$$\to_{\mathsf{L}}^{\mathsf{K}}\,\frac{\left\{\Gamma, \Delta \vdash T_{i,j} : A_{i,j}\right\}_{i \in \{1,\ldots,n\}, j \in \{1,\ldots,k_i\}} \quad \Gamma, \Delta, x_1 : \Box B_1, \ldots, x_n : \Box B_n \vdash M\,[x_1, \ldots, x_n, \vec{z} / y_1, \ldots, y_n, \vec{w}]_{\blacksquare} : \Box C}{\Gamma, \underbrace{f_1 : (A_{1,1}, \ldots, A_{1,k_1}) \to \Box B_1, \ldots, f_n : (A_{n,1}, \ldots, A_{n,k_n}) \to \Box B_n}_{\Delta} \vdash M\,[N_1, \ldots, N_n, \vec{z} / y_1, \ldots y_n, \vec{w}]_{\blacksquare} : \Box C}\,\dagger,\S$$

$* :=$ σ permutation over $\{1, \ldots, n\}$ $\quad$ $\star :=$ $FV(M) = \{x_1, \ldots x_n\}$ and $y_1, \ldots, y_n$ fresh

$\S :=$ each $N_i = f_i T_{i,1} \cdots T_{i,k_i}$ for $i \in \{1, \ldots, n\}$ $\quad$ $\dagger :=$ Γ contains no formula of the shape $(A_1 \cdots A_n) \to \Box B$

Fig. 5. Typing rules of the typing system CK^{F}.

- if $M = PQ \in \mathsf{nf}_{\beta\eta\kappa}(\Lambda)$, then both P and Q are in $\mathsf{nf}_{\beta\eta\kappa}(\Lambda)$ and there is a derivable type assignment $\Gamma \vdash M : C$, and derivable type assignments $\Gamma \vdash P : A \to C$ and $\Gamma \vdash Q : A$. We have that no $\rightsquigarrow_{\eta}$-rule can be applied to C because $M \in \mathsf{nf}_{\eta}(\Lambda)$; thus C must be atomic. We know that P cannot be in Λ^{λ} since $M \in \mathsf{nf}_{\beta}(\Lambda)$ and P cannot be in $\Lambda^{\blacksquare}$ because $\Gamma \vdash P : A \to C$ is derivable. Then by inductive hypothesis we have that $P = xT_1, \ldots T_n$ for some $T_1, \ldots, T_n \in \widehat{\Lambda}$. We conclude that $PQ \in \widehat{\Lambda}$;
- if $M = P\,[Q_1, \ldots, Q_n / x_1, \ldots, x_n]_{\blacksquare} \in \mathsf{nf}_{\beta\eta\kappa}(\Lambda)$, then there is a derivable type assignment $x_1 : B_1, \ldots, x_n : B_n \vdash P : C$ and derivable type assignments $\Gamma \vdash Q_i : \Box B_i$ for all $i \in \{1, \ldots, n\}$. Since $M \in \mathsf{nf}_{\beta\eta\kappa}(\Lambda)$, then no $\rightsquigarrow_{\beta\eta\kappa}$-rule can be applied to M, nor to P; thus $P \in \mathsf{nf}_{\beta\eta\kappa}(\Lambda)$. Similarly, since $M \in \mathsf{nf}_{\beta\eta\kappa}(\Lambda)$, then $Q_i \notin \Lambda^{\blacksquare}$ (otherwise we could apply $\rightsquigarrow_{\beta}^{2}$), $Q_i \in \mathsf{nf}_{\beta\kappa}(\Lambda)$ (since no $\rightsquigarrow_{\beta\kappa}$-rule can be applied to Q_i) and Q_i cannot be in $\mathsf{nf}_{\eta}(\Lambda)$ (because $Q_i : \Box B_i$ and otherwise $\rightsquigarrow_{\eta}$-steps could be applied on M) for all $i \in \{1, \ldots, n\}$. We conclude that $M \in \widehat{\Lambda}$.

4 A Canonical Type System for CK

In this section we present an alternative typing system for modal λ-terms where each term in $\widehat{\Lambda}$ admits exactly one typing derivation. The rules of this system (we call CK^{F}) are provided in Figure 5 and are conceived to reduce the non-determinism of the typing process, following the same approach used in designing focused sequent calculi [7,41,11]. Derivations and derivability in CK^{F} are defined analogously to Definition 1, using rules in CK^{F} instead of rules in $\mathsf{ND}_{\mathsf{CK}}$. We remark that the structural rules of weakening and contraction are admissible in the system.

We can now prove a result of *canonicity* of CK^{F} with respect to typing derivations of modal λ-terms in $\mathsf{nf}_{\beta\eta\kappa}(\Lambda)$.

Theorem 3. *Let $T \in \widehat{\Lambda}$ and $\Gamma \vdash T : A$ be a derivable type assignment. Then there is a unique (up to* ex*-rules) derivation of $\Gamma \vdash T : A$ in* CK^{F}.

Proof. The proof of this theorem follows from the correspondence between the inductive definition of terms in $\widehat{\Lambda}$ (definition 4) and the shape of the typing rules of CK^{F}.

By definition of $\widehat{\Lambda}$, we have the following cases:

- if $M = x$ is a variable, then $A = a$ must be an atomic formula. Since the sequent $\Gamma \vdash x : a$ is derivable in $\mathsf{ND}_{\mathsf{CK}}$, then it can only be the conclusion of a Id-rule if $x : a \in \Gamma$. We conclude since the rule Id is also in CK^{F} and such a derivation is unique;
- if $T = xN_1 \cdots N_m$, then $x : C = (A_1, \ldots, A_n) \to A$ is in Γ and $N_1, \ldots, N_m \in \widehat{\Lambda}$ with size smaller than $|M|$. By inductive hypothesis, for each $i \in \{1, \ldots, m\}$ the sequent $\Gamma \vdash N_i : A_i$ is derivable in $\mathsf{ND}_{\mathsf{CK}}$, therefore admits a unique derivation $\mathcal{D}_i$ in CK^{F}. We conclude since we have a unique derivation of $\Gamma \vdash M : A$ starting with a by $\to_{\mathsf{L}}^{\mathsf{ax}}$ whose premises are the conclusions $\mathcal{D}_1, \ldots, \mathcal{D}_n$;
- if $T = \lambda x_1^{A_1} \cdots \lambda x_n^{A_n}.M$ for a $M \neq \lambda y.N$, then $A = (A_1, \ldots, A_n) \to C$ for some types $A_1, \ldots, A_n, C$. Applying n times the rule Abs we know that $\Gamma \vdash \lambda x^{A_1} \ldots \lambda x^{A_n}.M : (A_1, \ldots, A_n) \to C$ iff $\Gamma, x : A_1, \ldots, x_n : A_n \vdash M : C$. By induction, we know that there is a unique derivation $\mathcal{D}_1$ of the sequent $\Gamma, x : A_1, \ldots, x_n : A_n \vdash M : C$ in CK^{F}. We conclude since we have a unique derivation of $\Gamma \vdash M : A$ starting with a by $\to_{\mathsf{R}}^{*}$ whose premise is the conclusion of $\mathcal{D}_1$;
- if $T = M\left[N_1, \ldots, N_n / y_1, \ldots, y_n\right]_{\blacksquare}$, then, we have two cases:
  - $N_i = x_i$ is a variable for all $i \in \{1, \ldots, n\}$. In this case a $\mathsf{ND}_{\mathsf{CK}}$ derivation of $\Gamma \vdash x_i : \Box A_i$ can only be made of a single Id-rule. This implies that $x_i : \Box A_i \in \Gamma$ for all $i \in \{1, \ldots, n\}$; thus $\Gamma = \Delta, x_1 : \Box A_1, \ldots x_n : \Box A_n$ for a context Δ. Moreover we must have a $\mathsf{ND}_{\mathsf{CK}}$-derivation of $x_1 : A_1, \ldots, x_n : A_n \vdash M : C$ for a C is such that $\Box C = A$; thus, since $|M| < |T|$, there is a unique derivation $\mathcal{D}_1$ of this latter sequent in CK^{F}. We conclude since we have a unique derivation of $\Gamma \vdash M : A$ starting with a by $\mathsf{K}^{\Box}$ whose premise is the unique derivation of $\Delta, x : A_1, \ldots, x_n : A_n \vdash M : C$ in CK^{F};
  - there are some $i \in \{1, \ldots, n\}$ such that N_i is of the form $f_i T_{i,1} \cdots T_{i,k_i}$ with $f_i : (A_{i,1}, \ldots, A_{i,k_i}) \to \Box B_i$ and $T_{i,1}, \ldots, T_{i,k_i} \in \widehat{\Lambda}$. For any $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k_i\}$ the sequent $\Gamma \vdash T_{i,j} : A_{i,j}$ is derivable in $\mathsf{ND}_{\mathsf{CK}}$, then, since the weakening rule is admissible, also $\Gamma, f_1 : (A_{1,1}, \ldots, A_{1,k_1}) \to \Box B_1, \ldots, f_n : (A_{n,1}, \ldots, A_{n,k_n}) \to \Box B_n \vdash N_{i,j} : A_{ij}$ is derivable. Since $|N_{i,j}| < |M|$, we can conclude by induction the existence of a unique derivation $\mathcal{D}_{i,j}$ in CK^{F} for this latter sequent. By similar argument, there is also a unique derivation for $\Gamma, \Delta, x_1 : \Box B_1, \ldots, x_n : \Box B_n \vdash M\left[x_1, \ldots, x_n, \vec{z} / y_1, \ldots, y_n, \vec{w}\right]_{\blacksquare} : \Box C$ allowing us to conclude the existence of a unique derivation of $\Gamma \vdash M : A$ starting with a by $\to_{\mathsf{L}}^{\mathsf{K}}$ whose rightmost premise is the conclusion of $\mathcal{D}'$ and whose other premises are the derivations $\mathcal{D}_{i,j}$ with $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k_i\}$;

5 Game Semantics for CK

In this section we recall definitions and results on the winning innocent strategies for the logic CK defined in [4]. For this purpose, we first recall the construction extending Hyland-Ong arenas [28,43] for intuitionistic propositional formulas to represent formulas containing modalities, and then we recall the characterization of the winning innocent strategies representing proofs in CK. We conclude by proving the full-completeness result between for those strategies by showing a one-to-one correspondence between strategies for type assignments of terms in normal forms and their (unique) typing derivations in CK^{F}.

5.1 Arenas with Modalities

We recall the definition of arenas with modalities from [4] extending the encoding of arenas from [29,25]. For this purpose, we assume the reader familiar with the definition of *two-color directed graph* (or *2-dag's* for short), i.e., directed acyclic graphs with two disjoint sets of directed edges $\rightarrow$ and $\rightsquigarrow$ (details can be found in [4,25]).

Definition 5. *The* arena *of a formula F is the 2-dag $[\![F]\!]$ with vertices are labeled by elements in $\mathcal{L} = \mathcal{A} \cup \{\Box\}$ inductively defined as follows:*

$$[\![a]\!] = a \qquad [\![A \to B]\!] = [\![A]\!] \twoheadrightarrow [\![B]\!] \qquad [\![\Box A]\!] = \Box \rightsquigarrow [\![A]\!] \tag{7}$$

where a and $\Box$ denote the graphs consisting of a single vertex labeled by a and $\Box$ respectively, and where the binary operation $\twoheadrightarrow$ and $\rightsquigarrow$ on 2-dag's are defined as follows:

$$\mathcal{G} \twoheadrightarrow \mathcal{H} = \langle\, V_{\mathcal{G}} \uplus V_{\mathcal{H}} \,,\, \overset{\mathcal{G}\uplus\mathcal{H}}{\rightarrow} \cup (\vec{R}_{\mathcal{G}} \curvearrowright \vec{R}_{\mathcal{H}}) \,,\, \overset{\mathcal{G}\uplus\mathcal{H}}{\rightsquigarrow} \,\rangle \quad \textit{and} \quad \mathcal{G} \rightsquigarrow \mathcal{H} = \langle\, V_{\mathcal{G}} \uplus V_{\mathcal{H}} \,,\, \overset{\mathcal{G}\uplus\mathcal{H}}{\rightarrow} \,,\, \overset{\mathcal{G}\uplus\mathcal{H}}{\rightsquigarrow} \cup (\vec{R}_{\mathcal{G}} \curvearrowright \vec{R}_{\mathcal{H}}) \,\rangle \quad \textit{with}$$

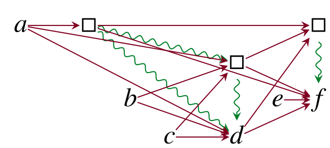
$$\begin{aligned} V_{\mathcal{G}} \uplus V_{\mathcal{H}} &= \{(v_i, i) \mid i \in \{0,1\} \textit{ and } v_0 \in V_{\mathcal{G}} \textit{ and } v_1 \in V_{\mathcal{H}}\} \quad \textit{and} \quad \ell((v_i, i)) = \ell(v_i) \\ \overset{\mathcal{G}\uplus\mathcal{H}}{\curvearrowright} &= \left\{((v_i, i), (w_i, i)) \mid i \in \{0,1\} \textit{ and } (v_0, w_0) \in \overset{\mathcal{G}}{\curvearrowright} \textit{ and } (v_1, w_1) \in \overset{\mathcal{H}}{\curvearrowright}\right\} \quad \textit{for each} \curvearrowright \in \{\rightarrow, \rightsquigarrow\} \\ (\vec{R}_{\mathcal{G}} \curvearrowright \vec{R}_{\mathcal{H}}) &= \left\{((v,0),(w,1)) \middle| v \in \vec{R}_{\mathcal{G}}, w \in \vec{R}_{\mathcal{H}}\right\} \quad \textit{where} \quad \vec{R}_X := \{v \in V_X \mid v \overset{X}{\rightarrow} w \textit{ for no } w \in V_X\} \end{aligned}$$

The arena of a sequent $A_1, \ldots, A_n \vdash C$ *is the arena* A *of* $[\![(A_1, \ldots, A_n) \to C]\!]$.

Remark 2. By construction, an arena $\mathcal{G}$ of a formula or a sequent $\Gamma \vdash C$ always admits a unique non $\Box$-labeled vertex in $\vec{R}_{\mathcal{G}}$, i.e., a unique vertex v with $\ell(v) \neq \Box$ such that there is no $w \in V_{\mathcal{G}}$ such that $v \overset{\mathcal{G}}{\rightarrow} w$.

We draw 2-dag's by representing a vertex v by its label $\ell(v)$. If v and w are vertices of an 2-dag, then we draw $v \rightarrow w$ if $v \rightarrow w$ and $v \rightsquigarrow w$ if $v \rightsquigarrow w$. By means of example, consider the arena below.

$$[\![(a \to \Box(b \to (c \to \Box d))) \to \Box(e \to f)]\!] \quad = \tag{8}$$



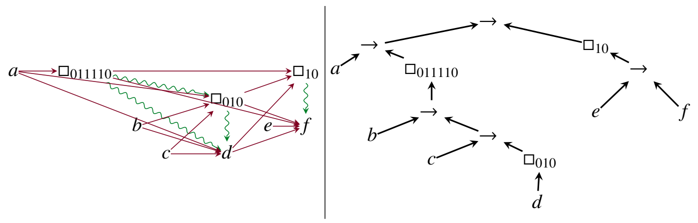
Remark 3. All arenas of the form $[\![(A_{\sigma(1)}, \ldots, A_{\sigma(n)}) \to C]\!]$ have the same representation for any σ permutation over $\{1, \ldots, n\}$. More in general, it can be shown that the arena of any two equivalent formulas modulo Currying $A \to (B \to C) \sim B \to (A \to C)$ can be depicted by the same arena. However, whenever there may be ambiguity because of the presence of two vertices with the same label, we may represent the vertex $v = ((\cdots(v', i_1)\cdots), i_n)$ (where $i_1, \ldots, i_n \in \{0,1\}$) by $\ell(v)_{i_1,\ldots,i_n}$ instead of simply $\ell(v) = \ell(v')$ (see Example 2).

Definition 6. *Let $[\![F]\!]$ be an arena and v one of its vertices. The* depth *of v is the number $d(v)$ of vertices in a $\rightarrow$-path from v to a vertex in $\vec{R}_{[\![F]\!]}$* [6]. *The* address *of v is defined as*

[6] As proven in [25,5], arenas are *stratified*, that is, all the $\rightarrow$-path from a vertex v to any vertex in $\vec{R}_{[\![F]\!]}$ have the same length. Therefore the number $d(v)$ is well-defined.

the unique sequence of modal vertices $\mathsf{add}(v) = m_1, \ldots, m_h$ *in* $V_{[\![F]\!]}$ *corresponding to the sequence of modalities in the path in the formula tree of* F *connecting the node of* v *to the root. If* $\mathsf{add}(v) = m_1, \ldots, m_h$*, we denote by* $\mathsf{add}^k(v) = m_k$ *its* k^{th} *element and we call the* height *of* v *(denoted* h_v*) the number of elements in* $\mathsf{add}(v)$*.*

Example 2. Below an alternative representation of its arena of the formula $(a \to \Box(b \to (c \to \Box d))) \to \Box(e \to f)$ in Equation (8) where the ambiguity of the vertex representation is avoided by the use of indices, the corresponding formula-tree, and the complete list of the addresses of all vertices in this arena.



$\mathsf{add}(a)$	$=\epsilon$
$\mathsf{add}(\Box_{011110})$	$=\epsilon$
$\mathsf{add}(b)$	$=\Box_{011110}$
$\mathsf{add}(c)$	$=\Box_{011110}$
$\mathsf{add}(\Box_{010})$	$=\Box_{011110}$
$\mathsf{add}(d)$	$=\Box_{011110}\Box_{010}$
$\mathsf{add}(\Box_{10})$	$=\epsilon$
$\mathsf{add}(e)$	$=\Box_{10}$
$\mathsf{add}(f)$	$=\Box_{10}$

5.2 Games and Winning Innocent Strategies

In this subsection, we briefly recall the definitions of games and winning strategies from [4] required to make the paper self-contained. Note that differently from the previous works, we here include the additional information of the *pointer function* in the definition of views. This information is crucial for the results in Section 4 where we provide a one-to-one correspondence between our winning strategies and modal λ-terms.

Definition 7. *Let* A *be an arena. We call a* move *an occurrence of a vertex* v *of* A *with* $\ell(v) \neq \Box$*. The* polarity *of a move* v *is the parity of* $d(v)$*: a move is a* $\circ$-move *(resp. a* $\bullet$-move*) if* $d(v)$ *is even (resp. odd).*

A pointed sequence *in* A *is a pair* $\mathsf{p} = \langle \mathsf{s}, f \rangle$ *where* $\mathsf{s} = \mathsf{s}_0, \ldots, \mathsf{s}_n$ *is a finite sequences of moves in* A *and a* pointer function $f\colon \{1, \ldots, n\} \to \{0, \ldots, n-1\}$ *such that* $f(i) < i$ *and* $\mathsf{s}_i \overset{\mathsf{A}}{\to} \mathsf{s}_{f(i)}$*. The* length *of* p *(denoted* $|\mathsf{p}|$*) is defined as the length of* s*, that is,* $|\mathsf{p}| = n + 1$*. Note that we also use* ϵ *to denote the* empty pointed sequence $\langle \epsilon, \emptyset \rangle$*.*

Remark 4. It follows by definition of view that the player $\circ$ (resp. $\bullet$) can only play vertices whose $d(v)$ is even (resp. odd). For this reason, for each $v \in V_{\mathcal{G}}$ we write v° (resp. $v^\bullet$) if the parity of $d(v)$ even (resp. odd).

Note that the parity of a modality in the address of a move may not be the same as the parity of the move itself. By means of example, consider the vertex c in Example 2 which belongs in the scope of two modalities $\Box_{011110}$ and $\Box_{010}$ with odd parity.

Given two pointed sequences $\mathsf{p} = \langle \mathsf{s}, f \rangle$ and $\mathsf{p}' = \langle \mathsf{s}', f' \rangle$ in A, we write $\mathsf{p} \sqsubseteq \mathsf{p}'$ whenever s is a prefix of s' (thus $|\mathsf{s}| \leq |\mathsf{s}'|$) and $f(i) = f'(i)$ for all $i \in \{1, \ldots, |\mathsf{p}'|\}$ and we say that p is a *predecessor* of p' if $\mathsf{p} \sqsubset \mathsf{p}'$ and $|\mathsf{p}| = |\mathsf{p}'| - 1$.

Definition 8. *Let* A *be an arena. A* play *on* A *is a pointed sequence* $\mathsf{p} = \langle \mathsf{s}, f \rangle$ *such that, either* $\mathsf{s} = \epsilon$*, or* s_i *and* s_{i+1} *have opposite polarities for all* $i \in \{0, \ldots, |\mathsf{p}| - 1\}$*.*

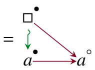
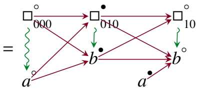
Arena	$[\![(\Box a) \to a]\!] =$ 	$[\![(\Box a \to \Box b) \to \Box(a \to b)]\!] =$ 
WIS	$\mathcal{S}_1 = \{\epsilon, a^\circ, a^\circ a^\bullet\}$	$\mathcal{S}_2 = \{\epsilon, b^\circ, b^\circ b^\bullet, b^\circ b^\bullet a^\circ, b^\circ b^\bullet a^\circ a^\bullet\}$
(failed) Derivation	$FAIL$ / $\Box a \vdash a$ / $\to^{\mathsf{R}}$: $\vdash \Box a \to a$	$FAIL$ / $\vdash a$ / $\mathsf{K}^\Box$: $\vdash \Box^\circ a$; ax: $b \vdash b$ / W: $b, a \vdash b$ / $\to^{\mathsf{L}}$: $b \vdash a \to b$ / $\mathsf{K}^\Box$: $\Box^\bullet b \vdash \Box^\circ(a \to b)$; $\to^{\mathsf{L}}$: $\Box^\circ a \to \Box^\bullet b \vdash \Box^\circ(a \to b)$ / $\to^{\mathsf{R}}$: $\vdash (\Box^\circ a \to \Box^\bullet b) \to \Box^\circ(a \to b)$

Fig. 6. Examples of WISs for arenas not corresponding to proofs.

The game of A *(denoted $\mathcal{G}_\mathsf{A}$) is the set of prefix-closed sets of plays over* A.

A view *is a play* $\mathsf{p} = \langle \mathsf{s}, f \rangle$ *such that either* $\mathsf{p} = \epsilon$ *or the following properties hold:*

- p *is* $\circ$-shortsighted *:* $f(2k) = 2k - 1$ *for every* $2k \in \{2, \ldots, |\mathsf{p}|\}$*;*
- p *is* $\bullet$-uniform *:* $\ell(\mathsf{s}_{2k+1}) = \ell(\mathsf{s}_{2k})$ *for every* $2k + 1 \in \{0, \ldots, |\mathsf{p}|\}$.

A winning innocent strategy *(or* WIS *for short) for the game* $\mathcal{G}_\mathsf{A}$ *is a finite non-empty prefix-closed set* $\mathcal{S}$ *of views in* $\mathcal{G}_\mathsf{A}$ *such that:*

- $\mathcal{S}$ *is* $\circ$-complete*: if* $\mathsf{p} \in \mathcal{S}$ *and* p *as odd length, then every successor of* p *(in* $\mathcal{G}_\mathsf{A}$*) is also in* $\mathcal{S}$ *;*
- p *is* $\bullet$-total*: if* $\mathsf{p} \in \mathcal{S}$ *and* p *has even length, then exactly one successor of* p *(in* $\mathcal{G}_\mathsf{A}$*) is in* $\mathcal{S}$ *;*

A view is maximal *in* $\mathcal{S}$ *if it is not prefix of any other view in* $\mathcal{S}$. $\mathcal{S}$ *is* trivial *if* $\mathcal{S} = \{\epsilon\}$. *We say that* $\mathcal{S}$ *is a* WIS *for a sequent* $A_1, \ldots, A_n \vdash C$ *if* $\mathcal{S}$ *is a* WIS *for* $[\![A_1, \ldots, A_n \vdash C]\!]$.

The definition of WIS above is a reformulation of the one in the literature of game semantics for intuitionistic propositional logic [28,13,25]. In presence of modalities, this definition requires to be refined to guarantee the possibility of gather modalities in *batches* corresponding to the modalities introduced by a single application of the $\mathsf{K}^\Box$ (see Figure 2). By means of example, consider the following arenas and their corresponding WISs, which cannot represent valid proofs in CK because of the impossibility of applying rules handling the modalities in a correct way.

Example 3. Consider the formulas $F_1 = (\Box a) \to a$ and $F_2 = (\Box a \to \Box b) \to \Box(a \to b)$ and their arenas in Figure 6. The set of views $\mathcal{S}_1$ and $\mathcal{S}_2$ are WISs for F_1 and F_2 respectively. However, these formulas are not provable in SCK because the proof search fails (see Figure 6). In particular, in the first case, no $\mathsf{K}^\Box$ can be applied because only there is a mismatch between the modalities on the left-hand side and on the right-hand side of the sequent; in the second case the problem is more subtle and, intuitively, is related to the fact that each $\mathsf{K}^\Box$ can remove only a single $\Box^\circ$ at a time, corresponding to the modality of the unique formula on the right-hand side of the sequent.

Therefore, in order to capture provability in CK, the notion of winning strategies has to be refined as follows.

Definition 9. *Let* $\mathsf{p} = (\mathsf{s}, f)$ *be a view in a strategy* $\mathcal{S}$ *on an arena* A*, and let* $h_\mathsf{p} = 1 + \max\{h_v \mid v \in \mathsf{p}\}$*. We define the* batched view *of* p *as the* $h_\mathsf{p} \times n$ *matrix* $\mathcal{F}(\mathsf{p}) = (\mathcal{F}(\mathsf{p})_0, \ldots, \mathcal{F}(\mathsf{p})_n)$ *with elements in* $V_\mathcal{G} \cup \{\epsilon\}$ *such that the each column* $\mathcal{F}(\mathsf{p})_i$ *is defined as follows:*

$$\mathcal{F}(\mathsf{p})_i = \begin{pmatrix} \mathcal{F}(\mathsf{p})_i^{h_\mathsf{p}} \\ \vdots \\ \mathcal{F}(\mathsf{p})_i^0 \end{pmatrix} \qquad \textit{where} \qquad \begin{cases} \mathcal{F}(\mathsf{p})_i^{h_\mathsf{p}} = \mathsf{add}^{h_{\mathsf{p}_i}}(\mathsf{p}_i), \ldots, \mathcal{F}(\mathsf{p})_i^{h_\mathsf{p} - h_{\mathsf{p}_i} + 1} = \mathsf{add}^1(\mathsf{p}_i) \\ \mathcal{F}(\mathsf{p})_i^{h_\mathsf{p} - h_{\mathsf{p}_i}} = \epsilon, \ldots, \mathcal{F}(\mathsf{p})_i^1 = \epsilon \\ \mathcal{F}(\mathsf{p})_i^0 = \mathsf{p}_i \end{cases}$$

We say that p *is* well-batched *if* $|\mathsf{add}(\mathsf{s}_{2k})| = |\mathsf{add}(\mathsf{s}_{2k+1})|$ *for every* $2k \in \{0, \ldots, |\mathsf{p}| - 1\}$*. Each well-batched view* p *induces an equivalence relation* $\overset{\mathcal{G}_\mathsf{p}}{\sim}$ *over* $V_\mathcal{G}$ *generated by:*

$$u \overset{\mathcal{G}_\mathsf{p}}{\sim}_1 w \quad \textit{iff} \quad u = \mathcal{F}(\mathsf{p})_{2k}^h \textit{ and } w = \mathcal{F}(\mathsf{p})_{2k+1}^h \textit{ for a } 2k < n-1 \textit{ and a } h \leq h_\mathsf{p} \tag{9}$$

A WIS $\mathcal{S}$ *is* linked *if it contains only well-batched views and if for every* $\mathsf{p} \in \mathcal{S}$ *the* $\overset{\mathcal{G}_\mathsf{p}}{\sim}$*-classes are of the shape* $\{v_1^\bullet, \ldots, v_n^\bullet, w^\circ\}$*.*

A CK-winning innocent strategy *(or* CK-WIS *for short) is a linked* WIS $\mathcal{S}$*.* [7]

Example 4. Consider the arenas in Figure 6. The batched view of the (unique) maximal views in $\mathcal{S}_1$ and $\mathcal{S}_2$ are $\begin{pmatrix} \epsilon & \Box^\bullet \\ a^\circ & a^\bullet \end{pmatrix}$ and $\begin{pmatrix} \Box^\circ_{10} & \Box^\bullet_{010} & \Box^\circ_{000} & \Box^\circ_{10} \\ b^\circ & b^\bullet & a^\circ & a^\bullet \end{pmatrix}$ respectively. The first is not well-batched because a° has height 0 while $a^\bullet$ has height 1, while the second, even if well-batched, is not linked because the $\overset{\mathcal{G}_\mathsf{p}}{\sim}$-class $\{\Box^\circ_{10}, \Box^\bullet_{010}, \Box^\circ_{000}\}$ contains two $\Box^\circ$.

The definition of CK-WISs allows us to obtain a full-completeness result with respect to CK which, together with the good compositionality properties of CK-WISs shown in [4,10], provides a full-complete denotational semantics for the logic CK. That is, every given CK-WIS is the encoding of a derivation in CK, and if a derivation $\mathcal{D}$ reduces via cut-elimination to a derivation $\mathcal{D}'$, then they are encoded by the same CK-WIS.

Theorem 4 ([4]). *The set of* CK-WIS*s is a full-complete denotational model for* CK*.*

5.3 Full Completeness for Modal Lambda Terms in Normal Form

We can prove the full completeness result using the type system CK^F and relying on Theorem 3. For this purpose, we have to extend the definition of α-equivalence from terms to type assignments in order to avoid technicality in our proofs, since in arenas we keep no track of variable names. For example, consider the α-equivalent terms $\lambda x.x$ and $\lambda y.y$ whose derivation should be considered non-equivalent due to the fact that α-equivalence does not extends to type assignments, therefore the two occurrence of the axiom rule with conclusion $x : a \vdash x : a$ and $y : a \vdash y : a$ should be considered distinct.[8]

[7] We here provide a simpler definition of CK-WISs w.r.t. the one in [4]. In fact, we are able here to simplify this definition because we are considering the $\Diamond$-free fragment of CK.

[8] Note that another possible way to deal with this problem is to label non-modal vertices of arenas by pairs of propositional atoms and variables instead of propositional variables only.

$$\left\{\!\!\left\{\mathsf{ax}\,\frac{}{\Gamma, x : c^\bullet \vdash x : c^\circ}\right\}\!\!\right\} = \{\epsilon, c^\circ, c^\circ c^\bullet\} \qquad \left\{\!\!\left\{\to^*_\mathsf{R}\,\frac{\begin{matrix}\mathcal{D}' \\ \Gamma, x_1 : A_1, \dots, x_n : A_n \vdash M : C\end{matrix}}{\Gamma \vdash \lambda x_1^{A_1} \cdots x_n^{A_n}.M : (A_1, \dots A_n) \to C}\right\}\!\!\right\} = \{\!\!\{\mathcal{D}'\}\!\!\}$$

$$\left\{\!\!\left\{\to^\mathsf{ax}_\mathsf{L}\,\frac{\left\{\begin{matrix}\mathcal{D}_i \\ \Gamma, (A_1, \dots, A_n) \to c \vdash N_i : A_i\end{matrix}\right\}_{i \in \{1,\dots,n\}}}{\Gamma, y : (A_1, \dots, A_n) \to c \vdash yN_1 \cdots N_n : c}\right\}\!\!\right\} = \{\epsilon, c^\circ, c^\circ c^\bullet\} \cup \{c^\circ c^\bullet \mathsf{p} \mid \epsilon \neq \mathsf{p} \in \{\!\!\{\mathcal{D}_i\}\!\!\} \text{ for a } i \in \{1, \dots, n\}\}$$

$$\left\{\!\!\left\{\mathsf{ex}\,\frac{\begin{matrix}\mathcal{D}' \\ \sigma(\Gamma) \vdash M : C\end{matrix}}{\Gamma \vdash M : C}\right\}\!\!\right\} = \{f_\sigma(\mathsf{p}) \mid \mathsf{p} \in \{\!\!\{\mathcal{D}'\}\!\!\}\,,\, f_\sigma \text{ isomorphism between } [\![\Gamma \vdash M : C]\!] \text{ and } [\![\sigma(\Gamma) \vdash M : C]\!]\}$$

where $f_\sigma(\mathsf{p})$ is the view obtained by applying f_σ to each move in p (and updating its pointer accordingly)

$$\left\{\!\!\left\{\mathsf{K}^\Box\,\frac{\begin{matrix}\mathcal{D}' \\ x_1 : A_1, \dots, x_n : A_n \vdash M : C\end{matrix}}{\Delta, y_1 : \Box A_1, \dots, y_n : A_n \vdash M\,[x_1, \dots, x_n / y_1, \dots, y_n]_\blacksquare : \Box C}\right\}\!\!\right\} = \{\!\!\{\mathcal{D}'\}\!\!\}$$

$$\left\{\!\!\left\{\to^\mathsf{K}_\mathsf{L}\,\frac{\left\{\begin{matrix}\mathcal{D}_{i,j} \\ \Gamma, \Delta \vdash T_{i,j} : A_{i,j}\end{matrix}\right\}_{i \in \{1,\dots,n\}, j \in \{1,\dots,k_i\}} \quad \begin{matrix}\mathcal{D}_0 \\ \Gamma, \Delta, x_1 : \Box B_1, \dots, x_n : \Box B_n \vdash M\,[x_1, \dots, x_n, \vec{z} / y_1, \dots, y_n, \vec{w}]_\blacksquare : \Box C\end{matrix}}{\Gamma, \underbrace{f_1 : (A_{1,1}, \dots, A_{1,k_1}) \to \Box B_1, \dots, f_n : (A_{n,1}, \dots, A_{n,k_n}) \to \Box B_n}_{\Delta} \vdash M\,[N_1, \dots, N_n, \vec{z} / y_1, \dots y_n, \vec{w}]_\blacksquare : \Box C}\right\}\!\!\right\}$$

$$= \{\!\!\{\mathcal{D}_0\}\!\!\} \cup \left(\bigcup_{i \in \{1,\dots,n\}} \left\{c^\circ b_i^\bullet \mathsf{p} \mid \epsilon \neq \mathsf{p} \in \left\{\!\!\left\{\mathcal{D}_{i,j}\right\}\!\!\right\} \text{ for a } j \in \{1, \dots, k_i\}\right\}\right)$$

where c° (resp. $b_i^\bullet$) is the unique non-$\Box$ vertex in $\vec{R}_{[\![\Box C]\!]}$ (resp. in $[\![\Box B_i]\!]$).

Fig. 7. Rules to construct a CK-WIS from a type derivation in $\mathsf{CK^F}$. For reasons of readability, we assume there is an implicit map identifying the moves in the arenas of the type assignment in the premises with the moves in the arena of the type assignment in the conclusion. Note that c° and $c^\bullet$ are occurrences of the same atom c, but we have decorate them to improve readability.

Definition 10. *Let $A_1, \dots, A_n \vdash C$ be a sequent. We define $\Lambda(\Gamma \vdash C)$ as the set of terms M such that the typing derivation $x_1 : A_1, \dots, x_n : A_n \vdash M : C$ is derivable, that is,*

$$\Lambda(\Gamma \vdash C) = \{M \in \Lambda \mid x_1 : A_1, \dots, x_n : A_n \vdash M : C \text{ is derivable for some } x_1, \dots, x_n\}\ .$$

If $M, N \in \Lambda(\Gamma \vdash C)$, we define $M =_\alpha^{\Gamma;C} N$ as the smallest equivalence relation generated by the rule $\dfrac{M\{z_1, \dots, z_n / x_1, \dots, x_n\} = N\{z_1, \dots, z_n / y_1, \dots, y_n\}}{M =_\alpha^{\Gamma;C} N}$ *z_i is fresh.*

From now on, we consider derivations up the α-equivalence defined above, that is, we consider derivations up to renaming of the variables occurring in a typing context.

Lemma 6. *Let $\mathcal{S}$ be a non-trivial* CK-WIS *over the arena $[\![\Gamma \vdash C]\!]$. Then there is a canonically defined $T_\mathcal{S} \in \widehat{\Lambda} \cap \Lambda(\Gamma \vdash C)$ admitting a unique typing derivation in* $\mathsf{CK^F}$.

Proof. We define a term $T_\mathcal{S}$ and a derivation $\mathcal{D}_\mathcal{S}(T)$ in $\mathsf{CK^F}$ of the type assignment $\Gamma \vdash T_\mathcal{S} : C$ by induction on the lexicographic order over the pairs $(|\mathcal{S}|, |C|)$:

1. if $C = c$ is atomic, then $\mathcal{S}$ must contain the CK-WIS $\{\epsilon, c^\circ, c^\circ c^\bullet\}$ because c° is the unique sink of $[\![\Gamma \vdash C]\!]$ and $c^\bullet$ is the unique (by $\bullet$-totality) $\bullet$-move justified by the unique previous $\circ$-move in $\mathcal{S}$. Note that by the well-batched condition we must have $\mathsf{add}(c^\circ) = \mathsf{add}(c^\bullet) = \epsilon$. Then

(a) either $\mathcal{S} = \{\epsilon, c, cc\}$, then $\Gamma = \Delta, c$ for a sequent Δ such that no move in Δ occurs in $\mathcal{S}$ (because of $\circ$-completeness). In this case $T = x$ and

$$\mathcal{D}_{\mathcal{S}}(T) = \mathsf{ax}\,\frac{}{\Delta, x : c \vdash x : c}$$

(b) or, since $\mathcal{S}$ is prefix-closed and well-batched, $\mathcal{S} = \left(\{\epsilon, c^\circ, c^\circ c^\bullet\} \bigcup_{i=0}^{n} cc\mathcal{S}_i\right)$ for some CK-WISs $\mathcal{S}_i$ for a sequent $\Gamma \vdash A_i$ for each $i \in \{1, \ldots, n\}$. Then $\Gamma = \Delta, (A_0, \ldots, A_n) \to c$. In this case $T = yN_1 \cdots N_n$ and

$$\mathcal{D}_{\mathcal{S}}(T) = \;\to^{\mathsf{ax}}_{\mathsf{L}}\,\frac{\overset{\mathcal{D}_{\mathcal{S}_1}(N_1)}{\Delta, y : (A_1, \ldots, A_n) \to c \vdash N_1 : A_1} \quad \cdots \quad \overset{\mathcal{D}_{\mathcal{S}_n}(N_n)}{\Delta, y : (A_1, \ldots, A_n) \to c \vdash N_n : A_n}}{\Delta, y : (A_1, \ldots, A_n) \to c \vdash c}$$

2. if $C = (A_1, \ldots, A_n) \to B$, then $T = \lambda x_1 \cdots \lambda x_n.T'$ and

$$\mathcal{D}_{\mathcal{S}}(T) = \;\to^{*}_{\mathsf{R}}\,\frac{\overset{\mathcal{D}_{\mathcal{S}}(T')}{\Gamma, x_1 : A_1, \ldots, x_n : A_n \vdash T' : B}}{\Gamma \vdash T : (A_1, \ldots, A_n) \to B}$$

3. $C = \Box^\circ A$ is a $\Box$-formula, then, if a sink v of $[\![\Gamma]\!]$ occurs as a move in $\mathcal{S}$, then it mus be justified by a sink of $[\![\Box A]\!]$. Therefore, by the well-batched condition, v must be in the scope of a $\Box$. We have two cases:

(a) either $\Gamma = \Sigma, \Box\Delta$ and no move in Σ occurs in $\mathcal{S}$. In this case we have that $T = M\left[x_1, \ldots, x_n / y_1, \ldots, y_n\right]_{\blacksquare}$ and

$$\mathcal{D}_{\mathcal{S}}(T) = \mathsf{K}^{\Box}\,\frac{\overset{\mathcal{D}_{\mathcal{S}}(M)}{x_1 : A_1, \ldots, x_n : A_n \vdash M : C}}{\Delta, y_1 : \Box A_1, \ldots, y_n : A_n \vdash M\left[x_1, \ldots, x_n / y_1, \ldots, y_n\right]_{\blacksquare} : \Box C}$$

(b) or $\Gamma = \Delta, (A_{1,1}, \ldots A_{1,k_1}) \to \Box B_1, \ldots, (A_{n,1}, \ldots A_{n,k_n}) \to \Box B_n$ for some $k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n > 0$ and a sequent Δ such that if $\Box^\bullet D$ or $(A_1, \ldots, A_n) \to \Box^\bullet D$ is in Δ, then $\Box^\bullet \not\approx_{\mathcal{S}} \Box^\circ$. In this case, by similar reasoning of (1.1b), there are for some CK-WISs $\mathcal{S}_i$ for the sequent $\Gamma \vdash A_i$ for each $i \in \{1, \ldots, n\}$ and a CK-WIS $\mathcal{S}_0$ for the sequent $\Gamma, \Delta, \Box B_1, \ldots, \Box B_n \vdash \Box C$ such that $\mathcal{S} = \left(\mathcal{S}_0 \cup \left(\bigcup_{i=0}^{n} \sigma_i \mathcal{S}_i\right)\right)$. Therefore $T = M\left[x_1, \ldots, x_n, \vec{z} / y_1, \ldots, y_n, \vec{w}\right]_{\blacksquare}$ and $\mathcal{D}_{\mathcal{S}}(T)$ is the following derivation

$$\to^{\mathsf{K}}_{\mathsf{L}}\,\frac{\left\{\overset{\mathcal{D}_{\mathcal{S}_{i,j}}(T_{i,j})}{\Gamma, \Delta \vdash T_{i,j} : A_{i,j}}\right\}_{\substack{i \in \{1, \ldots, n\},\\ j \in \{1, \ldots, k_i\}}} \qquad \overset{\mathcal{D}_{\mathcal{S}_0}\left(M\left[x_1, \ldots, x_n, \vec{z} / y_1, \ldots, y_n, \vec{w}\right]_{\blacksquare}\right)}{\Gamma, \Delta, \Delta' \vdash M\left[x_1, \ldots, x_n, \vec{z} / y_1, \ldots, y_n, \vec{w}\right]_{\blacksquare} : \Box C}}{\Gamma, \Delta \vdash M\left[N_1, \ldots, N_n, \vec{z} / y_1, \ldots y_n, \vec{w}\right]_{\blacksquare} : \Box C}$$

where $\Delta = f_1 : (A_{1,1}, \ldots, A_{1,k_1}) \to \Box B_1, \ldots, f_n : (A_{n,1}, \ldots, A_{n,k_n}) \to \Box B_n$ and $\Delta' = x_1 : \Box B_1, \ldots, x_n : \Box B_n$.

Theorem 5. *There is a one-to-one correspondence between terms in $\widehat{\Lambda} \cap \Lambda(\Gamma \vdash C)$ and* CK-WIS *for $\Gamma \vdash C$.*

Proof. Lemma 6 ensures that one CK-WIS $\mathcal{S}$ for $\Gamma \vdash C$, we can define a (unique) typing derivation $\mathcal{D}_{\mathcal{S}}$ in CK^{F} of a term $T_{\mathcal{S}} \in \widehat{\Lambda} \cap \Lambda(\Gamma \vdash C)$.

Conversely, given a type assignment $\Gamma \vdash T : C$ for a $T \in \widehat{\Lambda}$, then, we can uniquely define is a derivation $\mathcal{D}_T$ in CK^{F}. Thus, by Theorem 3, the type assignment $\Gamma \vdash T : C$ is derivable. We define $\mathcal{S}_T$ as the CK-WIS defined by induction on the number of rules in $\mathcal{D}_T$ using the rules in Figure 7.

We conclude since we have that $\mathcal{S}_{T_{\mathcal{S}}} = \mathcal{S}$ and $T_{\mathcal{S}_T} = T$ by definition.

6 Conclusion

In this paper we introduced a new modal λ-calculus for the $\Diamond$-free fragment of the constructive modal logic CK (without conjunction or disjunction). This lambda calculus builds on the work in [31], by adding a restricted η-reduction as well as two new reduction rules dealing with the explicit substitution constructor used to model the modality $\Box$. We proved normalization and confluence for this calculus and we provide a one-to-one correspondence between the set of terms in normal form and the set of winning strategies for the logic CK introduced in [4].

We foresee the possibility of extending the result presented in this paper to the entire disjunction-free fragment of CK, for which winning strategies are already defined in [4] are a fully complete denotational semantics. For this purpose, we should consider additional term constructors for terms whose type is a conjunction, as well as a new Let-like operator to model terms whose type is the modality $\Diamond$-formula similar to the one proposed in [9]. For this reason, in future works we plan to reformulate our lambda-calculus in the light of the novel line of research on calculi with explicit substitutions [33,34,2,1]. This approach would allow us to simplify some of the technicalities and achieve a more elegant operational semantics. Another interesting prospective is to extend our approach to operational semantics to the Fitch-style modal λ-calculus studied in [52].

At the same time, we plan to make explicit that our game semantics provides a concrete model for the *cartesian closed categories* provided with a *strong monoidal endofunctor* [9,32]. Indeed, categorical semantics of the calculus in [9] is modeled by means of *cartesian closed categories* equipped with a *strong monoidal endofunctor* taking into account the proof-theoretical behavior of the $\Box$-modality. We further conjecture that the syntactic category obtained via the quotient of modal terms modulo the relations we introduce in this paper is indeed a *free cartesian closed category* on a set of atoms with a *strong monoidal endofunctor*.

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