Infinitary cut-elimination via finite approximations

Matteo Acclavio[1]

Gianluca Curzi[2]

Giulio Guerrieri[1]

University of Sussex

University of Southern Denmark

VILLUM FONDEN

"X-IDF: Explainable Internet Data Flows"

University of Gothenburg

University of Birmingham

University of Sussex

Aix*Marseille université Socialement engagée

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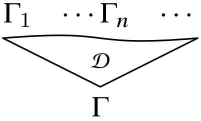
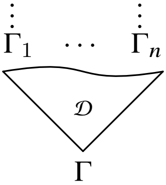
Infinite Proofs

Sequent calculus derivation: tree constructed using sequent calculus rules

0-ary	1-ary	2-ary		n-ary	ω-rule
$\mathsf{r}_0\,\dfrac{}{\Gamma}$	$\mathsf{r}_1\,\dfrac{\Gamma_1}{\Gamma}$	$\mathsf{r}_2\,\dfrac{\Gamma_1 \quad \Gamma_2}{\Gamma}$	$\cdots$	$\mathsf{r}_n\,\dfrac{\Gamma_1 \quad \cdots \quad \Gamma_n}{\Gamma}$	$\mathsf{r}_\omega\,\dfrac{\Gamma_1 \quad \cdots \quad \Gamma_n \quad \cdots}{\Gamma}$

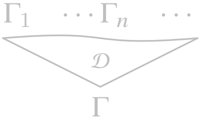
Infinte proofs

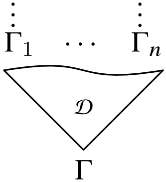




finite height	infinite height (non-wellfounded)
infinite width (infinite branching)	finite width
inductively defined	coinductively defined

Infinte proofs





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Coderivations

Formulas

$$A, B \coloneqq a \mid a^{\perp} \mid A \parr B \mid A \otimes B \mid\ !A \mid\ ?A$$

$$(A^{\perp})^{\perp} = A \qquad A \parr B^{\perp} = A^{\perp} \otimes B^{\perp} \qquad (!A)^{\perp} = ?A^{\perp}$$

Rules

$$\text{ax}\ \frac{}{A, A^{\perp}} \qquad \text{cut}\ \frac{\Gamma, A \quad A^{\perp}, \Delta}{\Gamma, \Delta} \qquad \parr\ \frac{\Gamma, A, B}{\Gamma, A \parr B} \qquad \otimes\ \frac{\Gamma, A \quad B, \Delta}{\Gamma, \Delta, A \otimes B}$$

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Not all infinite derivations are logically sound!

$$
\text{c!p}\,\dfrac{\begin{matrix}\mathcal{D}_0 \Vert \\ \Gamma, A\end{matrix} \qquad \text{c!p}\,\dfrac{\begin{matrix}\mathcal{D}_1 \Vert \\ \Gamma, A\end{matrix} \qquad \text{c!p}\,\dfrac{\begin{matrix}\mathcal{D}_n \Vert \\ \Gamma, A\end{matrix} \qquad \text{c!p}\,\dfrac{\vdots}{?\Gamma, !A}}{\cdots}\ \ \ddots}{?\Gamma, !A}}{?\Gamma, !A}
$$

$$
\text{cut}\,\dfrac{\text{ax}\,\dfrac{}{A^{\perp}, A} \qquad \text{cut}\,\dfrac{\text{ax}\,\dfrac{}{A^{\perp}, A} \qquad \text{cut}\,\dfrac{\vdots}{A}}{A}}{A}
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$$
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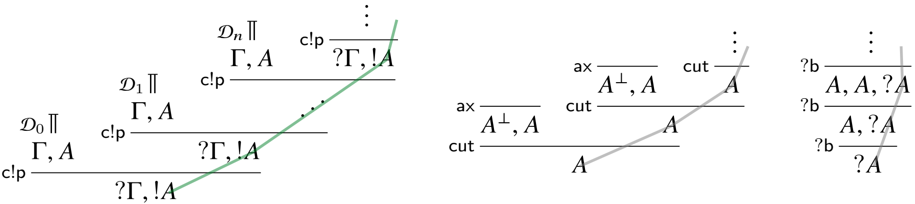
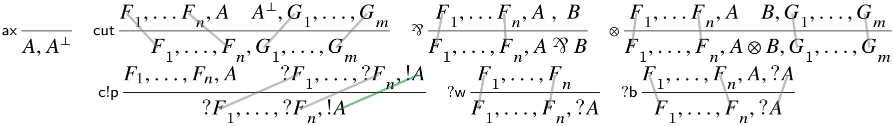
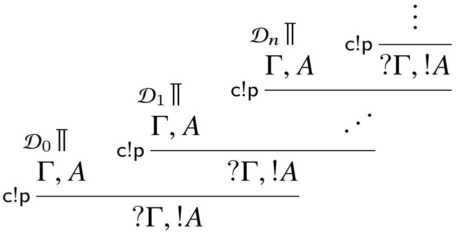


Progressiveness (ensuring correctness)

every infinite branch contains a progressing !-thread



- Regularity: only finitely many distinct sub-derivations (cyclic proofs)
- Weak-regularity: relax regularity allowing sub-derivations of the form



with $|\{\mathcal{D}_i\}_{i \in \mathbb{N}}|$ finite

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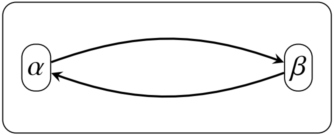
Non-wellfounded Box

Regular (possibly infinite) programs as proofs

Example (Token Ring)



$$\rightsquigarrow \quad \begin{array}{c}\pi_T \,\Vert \\ !\mathbf{B}\end{array} \quad := \quad \dfrac{\begin{array}{c}\pi_\alpha \,\Vert \\ \mathbf{B}\end{array} \qquad \dfrac{\begin{array}{c}\pi_\beta \,\Vert \\ \mathbf{B}\end{array} \qquad \begin{array}{c}\pi_T \,\Vert \\ !\mathbf{B}\end{array}}{!\mathbf{B}}\,\mathsf{c!p}}{!\mathbf{B}}\,\mathsf{c!p}$$

Cut-Elimination for infinite proofs

[finite derivations]

The standard strategy is to remove cut-rule “top-down”

[infinite derivations]

The standard strategy is to remove cut-rule “bottom-up”

$$\text{multicut}\ \frac{\Gamma_1, A_1 \quad A_1^\perp, \Gamma_2, A_2 \quad \cdots \quad A_n^\perp, \Gamma_n}{\Gamma_1, \ldots, \Gamma_n}$$

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My malaise with multicut:

$$\text{cut}\ \dfrac{\text{ax}\ \dfrac{}{A^{\perp}, A} \qquad \text{cut}\ \dfrac{\text{ax}\ \dfrac{}{A^{\perp}, A} \qquad \text{cut}\ \dfrac{\vdots}{\Gamma, A}}{\Gamma, A}}{\Gamma, A} \quad \longrightarrow^{*}_{\text{cut}} \quad \text{cut}_{\omega}\ \dfrac{\text{ax}\ \dfrac{}{A, A^{\perp}} \quad \cdots \quad \text{ax}\ \dfrac{}{A, A^{\perp}} \quad \cdots}{\Gamma, A}$$

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In this paper:
continuous cut-elimination as limit of (finitary) cut-elimination

$$\mathsf{cut}\,\dfrac{\mathsf{ax}\,\dfrac{}{A, A^\perp} \quad \Gamma, A}{\Gamma, A} \to_{\mathsf{cut}} \Gamma, A$$

$$\mathsf{cut}\,\dfrac{\parr\,\dfrac{\Gamma, A, B}{\Gamma, A \parr B} \quad \otimes\,\dfrac{\Delta, A^\perp \quad B^\perp, \Sigma}{\Delta, A^\perp \otimes B^\perp, \Sigma}}{\Gamma, \Delta, \Sigma} \to_{\mathsf{cut}} \mathsf{cut}\,\dfrac{\mathsf{cut}\,\dfrac{\Gamma, B, A \quad A^\perp, \Delta}{\Gamma, \Delta, B} \quad B^\perp, \Sigma}{\Gamma, \Delta, \Sigma}$$

$$\mathsf{cut}\,\dfrac{\mathsf{r}\,\dfrac{\Gamma_1, A}{\Gamma, A} \quad A^\perp, \Delta}{\Gamma, \Delta} \to_{\mathsf{cut}} \mathsf{r}\,\dfrac{\mathsf{cut}\,\dfrac{\Gamma_1, A \quad A^\perp, \Delta}{\Gamma_1, \Delta}}{\Gamma, \Delta}$$

$$\mathsf{cut}\,\dfrac{\mathsf{r}\,\dfrac{\Gamma_1, A \quad \Gamma_2}{\Gamma, A} \quad \Delta, A^\perp}{\Gamma, \Delta} \to_{\mathsf{cut}} \mathsf{r}\,\dfrac{\mathsf{cut}\,\dfrac{\Gamma_1, A \quad A^\perp, \Delta}{\Gamma_1, \Delta} \quad \Gamma_2}{\Gamma, \Delta}$$

$$\mathsf{cut}\,\dfrac{\mathsf{c!p}\,\dfrac{\Gamma, A \quad ?\Gamma, !A}{?\Gamma, !A} \quad ?\mathsf{w}\,\dfrac{\Delta}{\Delta, ?A^\perp}}{?\Gamma, \Delta} \to_{\mathsf{cut}} |\Gamma|\times?\mathsf{w}\,\dfrac{\Delta}{?\Gamma, \Delta}$$

$$\mathsf{cut}\,\dfrac{\mathsf{c!p}\,\dfrac{\Gamma, A \quad ?\Gamma, !A}{?\Gamma, !A} \quad ?\mathsf{b}\,\dfrac{\Delta, A^\perp, ?A^\perp}{\Delta, ?A^\perp}}{?\Gamma, \Delta} \to_{\mathsf{cut}} |\Gamma|\times?\mathsf{b}\,\dfrac{\mathsf{cut}\,\dfrac{?\Gamma, !A \quad \mathsf{cut}\,\dfrac{\Gamma, A \quad \Delta, A^\perp, ?A^\perp}{?\Gamma, \Delta, ?A^\perp}}{\Gamma, ?\Gamma, \Delta}}{?\Gamma, \Delta}$$

$$\mathsf{cut}\,\dfrac{\mathsf{c!p}\,\dfrac{\Gamma, A \quad ?\Gamma, !A}{?\Gamma, !A} \quad \mathsf{c!p}\,\dfrac{A^\perp, \Delta, B \quad ?A^\perp, ?\Delta, !B}{?A^\perp, ?\Delta, !B}}{?\Gamma, ?\Delta, !B} \to_{\mathsf{cut}} \mathsf{c!p}\,\dfrac{\mathsf{cut}\,\dfrac{\Gamma, A \quad A^\perp, \Delta, B}{\Gamma, \Delta, B} \quad \mathsf{cut}\,\dfrac{?\Gamma, !A \quad ?A^\perp, ?\Delta, !B}{?\Gamma, ?\Delta, !B}}{?\Gamma, ?\Delta, !B}$$

An **approximation** of a coderivation $\mathcal{D}$ is a coderivation obtained by pruning certain branches of the derivation tree.

Example

$$\text{c!p}\,\dfrac{\begin{matrix}\mathcal{D}_0 \Vert \\ \Gamma, A\end{matrix} \qquad \text{c!p}\,\dfrac{\begin{matrix}\mathcal{D}_1 \Vert \\ \Gamma, A\end{matrix} \qquad \text{c!p}\,\dfrac{\begin{matrix}\mathcal{D}_n \Vert \\ \Gamma, A\end{matrix} \qquad \text{c!p}\,\dfrac{\vdots}{?\Gamma, !A}}{\iddots}}{?\Gamma, !A}}{?\Gamma, !A} \quad \rightsquigarrow \quad \text{c!p}\,\dfrac{\text{hyp}\,\dfrac{}{\Gamma, A} \qquad \text{c!p}\,\dfrac{\begin{matrix}\mathcal{D}_1 \Vert \\ \Gamma, A\end{matrix} \qquad \text{hyp}\,\dfrac{}{?\Gamma, !A}}{?\Gamma, !A}}{?\Gamma, !A}$$

Theorem

The set of all approximations with the same conclusion is a Scott Domain.

- Cut-elimination strategy = *family* of sequences of coderivations

$$\{\mathcal{D} = \sigma_{\mathcal{D}}(0) \to_{\mathsf{cut}} \sigma_{\mathcal{D}}(1) \to_{\mathsf{cut}} \cdots\}_{\mathcal{D} \in \mathsf{pPLL}_2^\infty}$$

- Maximal = no cut-elimination can be applied to $\sigma_{\mathcal{D}}(\ell(\sigma_{\mathcal{D}}))$
- Cut-elimination function

$$f_\sigma(\mathcal{D}) = \bigsqcup_i (\text{cut-free initial segment of } \mathcal{D}_i)$$

Theorem

Let $\mathcal{D} \in \mathsf{pPLL}_2^\infty$. *Then* $f_{\sigma^\infty}(\mathcal{D}) \in \mathsf{pPLL}_2^\infty$.

Proof.

- Branches of $f_{\sigma^\infty}(\mathcal{D})$ as limits of "cut-free branches" of $\sigma_{\mathcal{D}}^\infty$;

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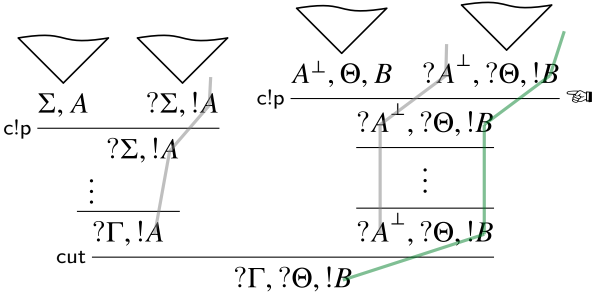
- It only suffices to look at the evolution of bottom-most cut-rules.



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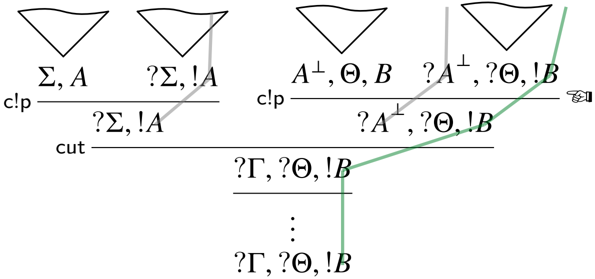

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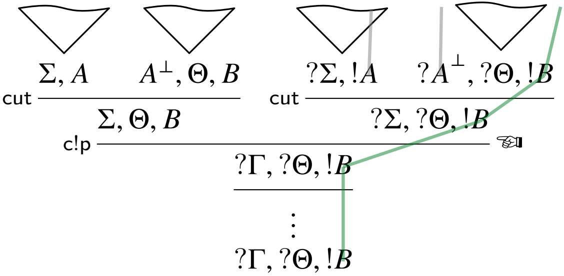
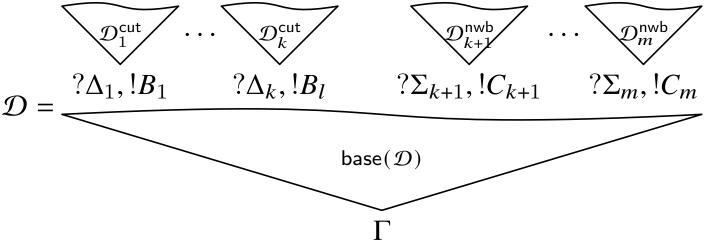


Theorem

If $\mathcal{D} \in \mathsf{pPLL}_2^\infty$ *is (weakly) regular, then so is* $f_{\sigma^\infty}(\mathcal{D})$.

Proof idea.



□

Lemma

If σ *and* σ' *are maximal cut-elimination functions, then* $f_\sigma = f_{\sigma'}$.

Conclusions and Future Works

Results:

- Notion of finite approximation for non-wellfounded derivations;
- New cut-elimination technique (no multicut);
- Relational semantics for infinite proofs;

Related / Future works:

- Non-uniform computations
  (Implicit Computational Complexity, Gianluca's talk at FICS)
- Apply the cut-elimination technique to other systems
  (e.g., modal-μ-calculus, PDL, ...)
- Session-types

Thank you

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Questions?