Exponentially Handsome Proof Nets

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Handsome proof nets were introduced by Retoré as a syntax for multiplicative linear logic. These proof nets are defined by means of cographs (graphs representing formulas) equipped with a perfect matching satisfying simple topological conditions. Hughes' combinatorial proofs – a proof system for classical logic able to capture proof equivalence – are defined as specific graph homomorphisms from a cographic proof net to a cograph.

Using the tools developed to formalize combinatorial proofs for modal logic, in this paper we extend Retoré's handsome proof nets to multiplicative linear logic with units and exponentials. This provides a sound and complete proof system, which captures a large portion of proof equivalence.

1 Introduction

One of the novelties introduced by linear logic [9] was the syntax of the proof nets. Proof nets are a graphical syntax [18, 19] for linear logic proofs, and capture proof equivalence for the multiplicative fragment of linear logic (denoted MLL). This means that if two MLL derivations are equivalent modulo rules permutations the two derivation can be encoded by the same proof net. The permutation considered are those switching the order of independent rules. Moreover, proof nets are a proof system (in the sense of [6]) for MLL since it is possible to check if a graph represents a correct derivation in polynomial time with respect to the size of the graph. This test can be conducted by means of a topological criterion, often refereed to as *correctness criterion* [7, 11, 27].

Several extensions of this syntax have been proposed to cover multiplicative linear logic with units ($\mathsf{MLL_u}$) or with exponentials (MELL), but none of them can be considered to be fully satisfactory. In presence of units, the correctness criterion requires to add additional edges, called *jumps*, to a proof net in order to associate each unit $\bot$ to an axiom or an unit 1 [15]. The same problem occurs in the exponential fragment due to the presence the weakening rule. The presence of the promotion rule in the exponential fragment poses an additional difficulty. Since the rule is context-sensitive, then the syntax requires the introduction of the so called *boxes* to delimit graph portions [25, 21, 22, 2, 1].

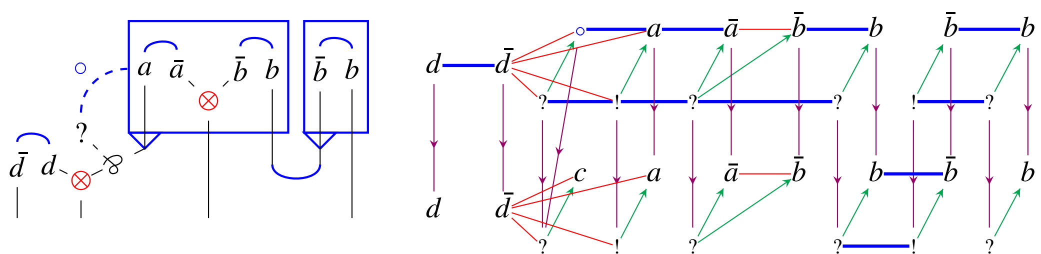

Figure 1: A proof net with jumps and its equivalent handsome proof net.

The quest for a satisfactory syntax extending proof nets for MLL has come to an end after [14], in which it is shown that it is not possible to have at the same time a syntax capturing the whole $\mathsf{MLL_u}$ proof equivalence and a polynomial correctness criterion, unless $\mathsf{P} = \mathsf{NP}$. The same problem arises in MELL.

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$$
\frac{}{a,\bar{a}}\,\mathsf{ax} \quad \frac{\Gamma,A,B}{\Gamma,A\parr B}\,\parr \quad \frac{\Gamma,A \quad B,\Delta}{\Gamma,A\otimes B,\Delta}\,\otimes \quad \frac{\Gamma}{\Gamma,\bot}\,\bot \quad \frac{}{1}\,1 \qquad \Bigg| \qquad \frac{\Gamma,A \quad \bar{A},\Delta}{\Gamma,\Delta}\,\mathsf{cut}
$$

$$
\frac{A,?\Gamma}{!A,?\Gamma}\,!p \quad \frac{\Gamma,A}{\Gamma,?A}\,\mathsf{der}_? \quad \frac{\Gamma}{\Gamma,?A}\,\mathsf{w}_? \quad \frac{\Gamma,?A,?A}{\Gamma,?A}\,\mathsf{c}_? \quad \frac{A,\Gamma}{!A,?\Gamma}\,\mathsf{s}!p \quad \frac{??A,\Gamma}{?A,\Gamma}\,\mathsf{dig}_?
$$

$$
\frac{}{a,\bar{a},\circ_1,\dots,\circ_n}\,\mathsf{ax}_j^n \quad \frac{}{1,\circ_1,\dots,\circ_n}\,1_j^n \quad \frac{\Gamma,\circ}{\Gamma,\bot}\,\bot^{\mathsf{j}} \quad \frac{\Gamma,\circ}{\Gamma,?A}\,\mathsf{w}^{\mathsf{j}}
$$

$$
\frac{\Gamma\{A\}}{\Gamma\{?A\}}\,\mathsf{der}_?^{\downarrow} \quad \frac{\Gamma\{??A\}}{\Gamma\{?A\}}\,\mathsf{dig}_?^{\downarrow} \quad \frac{\Gamma\{?A\parr ?A\}}{\Gamma\{?A\}}\,\mathsf{c}_?^{\downarrow} \quad \frac{\Gamma\{\circ\}}{\Gamma\{\bot\}}\,\bot^{\downarrow} \quad \frac{\Gamma\{\circ\}}{\Gamma\{?A\}}\,\mathsf{w}_?^{\downarrow}
$$

Figure 2: The Sequent calculus rules, the cut-rule and the deep inference rules for dereliction and digging, ?-contraction and ?-weakening

X	MLL	$\mathsf{MLL_u}$	MELL
$\mathsf{X}^{\mathsf{seq}}$	$\{\mathsf{ax},\parr,\otimes\}$	$\{\mathsf{ax},\parr,\otimes,1,\bot\}$	$\{\mathsf{ax},\parr,\otimes,!p,\mathsf{der}_?,\mathsf{w}_?,\mathsf{c}_?,1,\bot\}$
X^{j}	$\mathsf{X}^{\mathsf{seq}}$	$\{\mathsf{ax}_j,1_j,\parr,\otimes\}$	$\{\mathsf{ax}_j,1_j,\bot^{\mathsf{j}},\mathsf{w}^{\mathsf{j}},\parr,\otimes,\mathsf{s}!p,\mathsf{der}_?,\mathsf{dig}_?,\mathsf{c}_?\}$
X^{LL}	$\{\mathsf{ax},\parr,\otimes\}$	$\{\mathsf{ax}_j,1_j,\parr,\otimes\}$	$\{\mathsf{ax}_j,1_j,\parr,\otimes,\mathsf{s}!p\}$
$\mathsf{X}^{\downarrow}$	$\varnothing$	$\{\bot^{\downarrow}\}$	$\{\bot^{\downarrow},\mathsf{w}_?^{\downarrow},\mathsf{der}_?^{\downarrow},\mathsf{dig}_?^{\downarrow},\mathsf{c}_?^{\downarrow}\}$

Figure 3: Rules systems for MLL, $\mathsf{MLL_u}$ and MELL

In this paper we present a syntax for $\mathsf{MLL_u}$ and MELL by means of combinatorial proofs [16, 17, 28]. For this purpose, in Section 2 we prove a decomposition theorem for MELL using deep inference rules [12, 13, 5]. Thanks to this result, we are able to construct proofs with a *linear* part and a *resource management* part. In absence of units or exponentials, the linear part can be represented by Retoré's "handsome" proof nets [27]. These proof nets are cographs (graphs representing formulas) equipped with a perfect matching (representing axiom links). In Section 3 we define relation webs, which generalize cographs, to encode formulas with modalities. In Section 4 we extend Retoré's proof nets to relation webs with matching, called RGB-cographs, which encode the linear part of a MELL proof, i.e., axioms rules, (soft) promotions and logic connectives. In Section 5 we define the MELL fibrations which take care of representing the resource management part of our proofs, which is the part of the proof containing weakenings, contractions, derelictions and diggings. In Section 6 we define combinatorial proofs as MELL fibrations from an RGB-cograph to a relation web. Finally, in Section 7, we define handsome proof nets for MELL as compositions of combinatorial proofs by means of cut.

2 Sequent Calculus and Calculus of Structures

We define formulas in negation normal form generated from a countable set of propositional variables $\mathscr{A} = \{a,b,\dots\}$ and their duals $\bar{\mathscr{A}} = \{\bar{a},\bar{b},\dots\}$ by the following grammar:

$$
A,B ::= a \mid \bar{a} \mid A\parr B \mid A\otimes B \mid !A \mid ?A \mid \bot \mid 1 \mid \circ
$$

Linear negation $\bar{\cdot}$ is defined on formulas through the De Morgan laws: $\bar{\bar{A}} = A$, $\overline{A\otimes B} = \bar{A}\parr\bar{B}$, $\overline{!A} = ?\bar{A}$, $\bar{1} = \bot$, $\bar{\circ} = \circ$. A *sequent* $\Gamma = A_1,\dots,A_n$ is a non-empty multiset of formulas. The meaning of $\circ$ will be explained later in this section. Untill then, it can be interpreted as a placeholder.

In this paper we consider the consider the *multiplicative linear logic* and its extensions with *units* and

$$\dfrac{\Gamma',A \quad \dfrac{\Gamma'',B,C \quad \Gamma''',D}{\Gamma'',\Gamma''',B,C\otimes D}\otimes}{\Gamma,A\otimes B,C\otimes D)}\otimes \simeq \dfrac{\dfrac{\Gamma',A \quad \Gamma'',B,C}{\Gamma,A\otimes B,C}\otimes \quad \Gamma''',D}{\Gamma,A\otimes B,C\otimes D}\otimes \qquad \dfrac{\dfrac{\Gamma,\Delta,\Delta'}{\Gamma,A,\Delta'}\rho}{\Gamma,A,A'}\rho' \simeq \dfrac{\dfrac{\Gamma,\Delta,\Delta'}{\Gamma,\Delta,A'}\rho'}{\Gamma,A,A'}\rho \qquad \dfrac{\dfrac{\Gamma,?A}{\Gamma,?A,?A}\mathsf{w}_?}{\Gamma,?A}\mathsf{c}_? \simeq \Gamma,?A$$

$$\dfrac{\dfrac{\Gamma,?A,?A}{\Gamma,?A}\mathsf{c}_?}{\Gamma,?A,?A}\mathsf{w}_? \simeq \Gamma,?A,?A \qquad \dfrac{\dfrac{\Gamma,?A_1,?A_2,?A_3}{\Gamma,?A_1,?A}\mathsf{c}_?}{\Gamma,?A}\mathsf{c}_? \simeq \dfrac{\dfrac{\Gamma,?A_1,?A_2,?A_3}{\Gamma,?A,?A_3}\mathsf{c}_?}{\Gamma,?A}\mathsf{c}_? \qquad \dfrac{\dfrac{\Gamma',\Delta,B}{\Gamma',A,B}\tau \quad \Gamma'',C}{\Gamma',\Gamma'',A,B\otimes C}\otimes \simeq \dfrac{\dfrac{\Gamma',\Delta,B \quad \Gamma'',C}{\Gamma',\Gamma'',\Delta,B\otimes C}\otimes}{\Gamma',\Gamma'',A,B\otimes C}\tau$$

Figure 4: The proof equivalence $\simeq$ in MELL is defined by all $\rho,\rho',\tau \in \{\invamp,\bot,\mathsf{w}_?,\mathsf{c}_?,\mathsf{der}_?\}$.

exponentials[1] denoted respectively by MLL, $\mathsf{MLL_u}$ and MELL [9]. We define the sequent calculus rules in Figure 2 and a (cut-free) sequent system $\mathsf{X}^{\mathsf{seq}}$ for each $\mathsf{X} \in \{\mathsf{MLL},\mathsf{MLL_u},\mathsf{MELL}\}$ in the first three lines of Figure 3. We say that a formula F is provable in X (denoted by $\overset{\mathsf{X}}{\vdash} F$) if there is a derivation of F in the system $\mathsf{X}^{\mathsf{seq}}$.

In MELL, the rules permutations in Figure 4 generate the equivalence relation $\simeq$ between derivations. This notion of *proof equivalence* is needed to prove the cut-elimination theorem for MELL [9]. However, as shown in [14], the rule permutation in Figure 4 with $\tau = \bot$ makes the complexity of checking proof equivalence in $\mathsf{MLL_u}$ non-polynomial. The same argument applies to MELL for $\tau = \mathsf{w}_?$. As consequence, we cannot design a syntax $\mathscr{S}$ for MELL satisfying the two following desiderata:

- correctedness in $\mathscr{S}$ can be checked in polynomial time, i.e., we can check in polynomial time if an object expressed in the syntax $\mathscr{S}$ represents a correct proof in MELL;
- $\mathscr{S}$ captures proof equivalence, that is, if $[\![\pi]\!]$ and $[\![\pi']\!]$ are the encodings in $\mathscr{S}$ of two derivations π and π' in $\mathsf{MELL}^{\mathsf{seq}}$ and $\pi \simeq \pi'$, then $[\![\pi]\!] = [\![\pi']\!]$.

The complexity of checking proof equivalence depends on the fact that each $\mathsf{w}_?$ and $\bot$ can be assigned to an instance of ax or a 1 by permuting them upwards in a derivation (this assignation is called a *jump*). Since $\simeq$ allows to change such assignations, the equivalence check has to test all possible jumps, and the number of jumps is exponential with respect to the number of $\bot$- and $\mathsf{w}_?$-instances.

Since we cannot aspire to capture the whole proof equivalence $\simeq$, in order to have a polynomial correctedness criterion complexity in our syntax, we define the *fixed-jump equivalence* (denoted by $\simeq_{\mathsf{J}}$) by fixing in a derivation π a jump for each $\bot$- and $\mathsf{w}_?$-instance[2]. To keep track of jumps, we introduce the rules ax_j and 1_j, where $\mathsf{ax}_j = \{\mathsf{ax}_j^n \mid n \in \mathbb{N}\}$ and $1_j = \{1_j^n \mid n \in \mathbb{N}\}$, together with $\bot^{\mathsf{j}}$ and w^{j}. These rules allow to assign to each ax (or 1) a bunch of jump placeholders denoted by $\circ$, by using the rule ax_j. Each placeholder is further used by a single $\bot$- and $\mathsf{w}_?$-rule instance application. We define sequent calculi X^j for each $\mathsf{X} \in \{\mathsf{MLL},\mathsf{MLL_u},\mathsf{MELL}\}$ in Figure 3. In $\mathsf{MELL}^{\mathsf{j}}$ we replace the *promotion* rule $!p$ with the *soft promotion* rule [20] $\mathsf{s}!p$ together with the *digging* rule $\mathsf{dig}_?$. Soft promotion allows to group the ! introduced by a promotion with all the ? of its context formula.

Proposition 1. *If F is a fomula, then* $\overset{\mathsf{X}}{\vdash} F \iff \overset{\mathsf{X}^j}{\vdash} F$

Proof. By rules permutations, we can move each occurrence ρ of a $\mathsf{w}_?$- or a $\bot$-rule up in the derivation until it reaches the assigned occurrence σ_ρ of an ax_j- or a 1_j-rule. Then we replace σ_ρ with an occurrence of the same rule σ_ρ with an additional $\circ$ in the conclusion, and ρ with an occurrence of ρ^{j} applied to this

[1] In this paper, where not otherwise specified, we consider multiplicative linear logic with exponential including units.

[2] By mean of example, consider the derivations of $\Gamma = \bot, a, b, \bar{a} \otimes \bar{b}$ in $\mathsf{MLL_u}$. It admits only two possible jumps (one for each ax). All the possible derivations of Γ are $\simeq$-equivalent. However, only derivation with the same jump are $\simeq_{\mathsf{J}}$-equivalent.

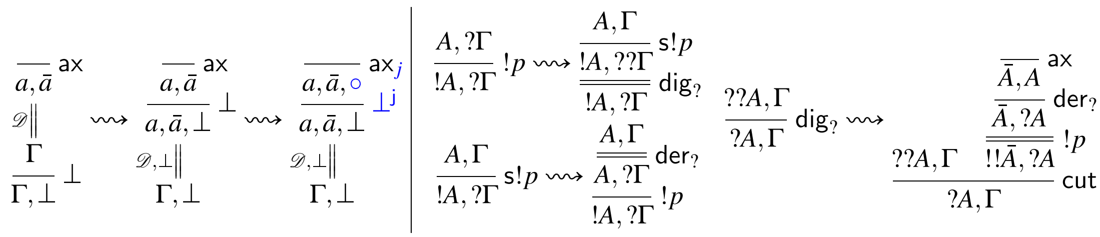


Figure 5: On the left: how to transform a $\bot$ to a $\bot^{\mathsf{j}}$. On the right: how to replace $!p$ with $\mathsf{s}!p$ and $\mathsf{dig}_?$ and vice versa.

fresh $\circ$ (an example is shown in Figure 5). Moreover, every $!p$ can be replaced by a $\mathsf{s}!p$ followed by a finite number of $\mathsf{dig}_?$ and vice versa by MELL cut-elimination theorem [9] (see Figure 5). $\square$

We call an instance of a $\mathsf{c}_?$ rule *jump-erasing* if (at least) one of the two contracted formulas has been introduced by a weakening rule.

Remark 2. If π is a derivation in $\mathsf{MELL}^{\mathsf{j}}$, then there is a derivation π' in $\mathsf{MELL}^{\mathsf{j}}$ such that $\pi \equiv \pi'$ and π' contains no jump-erasing contractions. Thus, every derivation in MELL is $\simeq_{\mathsf{J}}$-equivalent to a derivation in MELL containing no jump-erasing contractions.

The *deep inference* [12, 13, 5] rules for the systems in Figure 3 are defined in Figure 2. We denote by $\Gamma\{\ \}$ a *context*, which is a sequent or a formula with an "hole" in place of an atom. The use of deep rules allow us to prove the following decomposition theorem by pushing all $\mathsf{w}_?$, $\mathsf{c}_?$, $\mathsf{dig}_?$ and $\mathsf{der}_?$ inferences to the bottom of a derivation. We write $F' \overset{X}{\longmapsto} F$ if there is a derivation form F' to F using only rules in X.

Theorem 3. *If F is a formula, then $\overset{\mathsf{MELL}}{\longmapsto} F$ iff there is a formula F' such that $\overset{\mathsf{MELL}^{\mathsf{LL}}}{\longmapsto} F' \overset{\mathsf{MELL}^{\downarrow}}{\longmapsto} F$.*

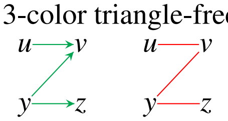
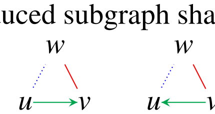
Proof. By Proposition 1, we consider a derivation $\overset{\mathsf{MELL}^{\mathsf{j}}}{\longmapsto} F$ and we replace each occurrence of $\bot^{\mathsf{j}}$, w^{j}, $\mathsf{dig}_?$, $\mathsf{der}_?$ and $\mathsf{w}_?$ by an occurrence of its deep version $\bot^{\downarrow}$, $\mathsf{w}_?^{\downarrow}$, $\mathsf{dig}_?^{\downarrow}$, $\mathsf{der}_?^{\downarrow}$ or $\mathsf{c}_?^{\downarrow}$ and then to push these inferences to the bottom of the derivation. The converse is proven similarly. $\square$

3 Relation Webs

A *directed graph* $\mathscr{G} = \langle V_{\mathscr{G}}, \overset{\mathscr{G}}{\rightharpoonup} \rangle$ is a set $V_{\mathscr{G}}$ of *vertices* equipped with a binary *edge relation* $\overset{\mathscr{G}}{\rightharpoonup} \subseteq V_{\mathscr{G}} \times V_{\mathscr{G}}$. An *undirected graph* $\mathscr{G} = \langle V_{\mathscr{G}}, \overset{\mathscr{G}}{\frown} \rangle$ is a graph whose *edge relation* $\overset{\mathscr{G}}{\frown} \subseteq V_{\mathscr{G}} \times V_{\mathscr{G}}$ is irreflexive and symmetric. A *mixed graph* is a triple $\mathscr{G} = \langle V_{\mathscr{G}}, \overset{\mathscr{G}}{\frown}, \overset{\mathscr{G}}{\rightharpoonup} \rangle$ where $\langle V_{\mathscr{G}}, \overset{\mathscr{G}}{\frown} \rangle$ is an undirected graph and $\langle V_{\mathscr{G}}, \overset{\mathscr{G}}{\rightharpoonup} \rangle$ is a directed graph, such that $\overset{\mathscr{G}}{\frown} \cap \overset{\mathscr{G}}{\rightharpoonup} = \emptyset$. If $V' \subset V_{\mathscr{G}}$, the *subgraph induced by* V' is the graph $\mathscr{G}|_{V'} = \langle V', \overset{\mathscr{G}}{\frown} \cap (V' \times V'), \overset{\mathscr{G}}{\rightharpoonup} \cap (V' \times V') \rangle$. We omit the index/superscript $\mathscr{G}$ when it is clear from the context. When drawing a graph we use v—w for $v \frown w$, and $v \rightarrow w$ for $v \rightharpoonup w$, and we use either $v \cdots w$ or draw no edge at all otherwise.

Definition 4. A *relation web* is a mixed graph $\mathscr{G} = \langle V_{\mathscr{G}}, \overset{\mathscr{G}}{\frown}, \overset{\mathscr{G}}{\rightharpoonup} \rangle$ such that:

- $\overset{\mathscr{G}}{\rightharpoonup}$ is transitive and irreflexive;
- $\mathscr{G}$ is Z-free and 3-color triangle-free, i.e., $\mathscr{G}$ does not contain an induced subgraph shape:

Z-freeness:  3-color triangle-freeness:  (1)

A *cograph* is a Z-free undirect graph.

Let $\mathcal{G}$ and $\mathcal{H}$ be two disjoint mixed graphs. We define the following operations:

$$\begin{array}{rcl} \mathcal{G} \parr \mathcal{H} & = & \langle V_{\mathcal{G}} \cup V_{\mathcal{H}} \, , \, \overset{\mathcal{G}}{\frown} \cup \overset{\mathcal{H}}{\frown} \, , \, \overset{\mathcal{G}}{\leadsto} \cup \overset{\mathcal{H}}{\leadsto} \rangle \\ \mathcal{G} \lhd \mathcal{H} & = & \langle V_{\mathcal{G}} \cup V_{\mathcal{H}} \, , \, \overset{\mathcal{G}}{\frown} \cup \overset{\mathcal{H}}{\frown} \, , \, \overset{\mathcal{G}}{\leadsto} \cup \overset{\mathcal{H}}{\leadsto} \cup \{(u,v) \mid u \in V_{\mathcal{G}}, v \in V_{\mathcal{H}}\} \rangle \\ \mathcal{G} \otimes \mathcal{H} & = & \langle V_{\mathcal{G}} \cup V_{\mathcal{H}} \, , \, \overset{\mathcal{G}}{\frown} \cup \overset{\mathcal{H}}{\frown} \cup \{(u,v),(v,u) \mid u \in V_{\mathcal{G}}, v \in V_{\mathcal{H}}\} \, , \, \overset{\mathcal{G}}{\leadsto} \cup \overset{\mathcal{H}}{\leadsto} \rangle \end{array} \tag{2}$$

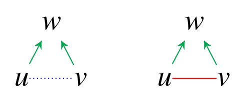
Theorem 5 ([**?**]). *A mixed graph is a relation web if and only if it can be constructed from single vertices using the three operations* $\parr$, $\lhd$ *and* $\otimes$ *defined in* (2).

A relation web is *labeled* if all its vertices carry a label selected from a label set $\mathcal{L}$. We write $l(v)$ for the label of v. For each formula F we define the labeled relation web $[\![F]\!]$ where the label set $\mathcal{L} = \mathcal{A} \cup \bar{\mathcal{A}} \cup \{!, ?, 1, \bot, \circ\}$. We use the notations $\bullet_a$, $\bullet_{\bar{a}}$, $!$, $?$, 1, $\bot$ and $\circ$ for the graph consisting of a single vertex that is labeled with a, $\bar{a}$, $!$, $?$, 1, $\bot$ and $\circ$ respectively.

$$\begin{array}{rclcrclcrclcrclcrcl} [\![a]\!] & = & \bullet_a & & [\![A \otimes B]\!] & = & [\![A]\!] \otimes [\![B]\!] & & [\![!A]\!] & = & ! \lhd [\![A]\!] & & [\![1]\!] & = & 1 & & & & \\ & & & & & & & & & & & & & & & & [\![\circ]\!] & = & \circ \\ [\![\bar{a}]\!] & = & \bullet_{\bar{a}} & & [\![A \parr B]\!] & = & [\![A]\!] \parr [\![B]\!] & & [\![?A]\!] & = & ? \lhd [\![A]\!] & & [\![\bot]\!] & = & \bot & & & & \end{array} \tag{3}$$

For a sequent $\Gamma = A_1, \ldots, A_n$ we define $[\![\Gamma]\!] = [\![A_1, \ldots, A_n]\!] = [\![A_1]\!] \parr \cdots \parr [\![A_n]\!]$.

Definition 6. A relation web $\mathcal{G}$ is *modalic* if for any vertices u, v, w with $u \leadsto w$ and $v \leadsto w$ we have $u \leadsto v$ or $v \leadsto u$ or $u = v$, i.e., $\mathcal{G}$ does not contain the two configurations below.

Forbidden configurations for modalic relation webs:  (4)

A labeled modalic relation web $\mathcal{G}$ is *properly labeled* if its label set is $\mathcal{L} = \mathcal{A} \cup \bar{\mathcal{A}} \cup \{!, ?, 1, \bot, \circ\}$, such that whenever there are v, w with $v \leadsto w$ then $l(v) \in \{!, ?\}$.

By adapting the proofs in [4], we have the following results:

Theorem 7. *A relation web is the translation of a formula iff it is modalic and properly labeled.*

Proposition 8. *For two formulas* F *and* F'*, we have* $[\![F]\!] = [\![F']\!]$ *iff* F *and* F' *are equivalent modulo associativity of and commutativity of* $\otimes$ *and* $\parr$.

4 RGB-cographs

Definition 9. An *RGB-cograph* is a tuple $\mathcal{G} = \langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\frown}, \overset{\mathcal{G}}{\leadsto}, \overset{\mathcal{G}}{\curlyvee} \rangle$ where $\mathcal{G}_{[\![]\!]} = \langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\frown}, \overset{\mathcal{G}}{\leadsto} \rangle$ is a modalic relation web, $V_{\mathcal{G}}$ is the disjoint union of five sets $V_{\mathcal{G}}^{\bullet} \uplus V_{\mathcal{G}}^{1} \uplus V_{\mathcal{G}}^{\circ} \uplus V_{\mathcal{G}}^{!} \uplus V_{\mathcal{G}}^{?}$, and $\overset{\mathcal{G}}{\curlyvee}$ is an equivalence relation over $V_{\mathcal{G}}$, called the *linking*, such that

- if $v \in V_{\mathcal{G}}^{\bullet} \cup V_{\mathcal{G}}^{1} \cup V_{\mathcal{G}}^{\circ}$ then there is no $w \in V_{\mathcal{G}}$ such that $v \leadsto w$;
- if $v \curlyvee w$ then $v, w \in V_{\mathcal{G}}^{\bullet} \cup V_{\mathcal{G}}^{\circ}$ or $v, w \in V_{\mathcal{G}}^{1} \cup V_{\mathcal{G}}^{\circ}$ or $v, w \in V_{\mathcal{G}}^{!} \cup V_{\mathcal{G}}^{?}$;
- if $v \in V_{\mathcal{G}}^{\bullet}$ then there is exactly another $w \in V_{\mathcal{G}}^{\bullet}$ with $v \curlyvee w$ and $v \neq w$;
- if $v \in V_{\mathcal{G}}^{\circ}$ then there is a $u \in V_{\mathcal{G}}^{\bullet} \cup V_{\mathcal{G}}^{1}$ such that $w \curlyvee v$;
- if $v \in V_{\mathcal{G}}^{1}$ then $w \in V_{\mathcal{G}}^{\circ}$ for all $w \curlyvee v$;
- if $v \in V_{\mathcal{G}}^{!} \cup V_{\mathcal{G}}^{?}$ then there is a unique $w \in V_{\mathcal{G}}^{!}$ such that $w \curlyvee v$;

The vertices in $V^{\bullet}_{\mathcal{G}}$, $V^{1}_{\mathcal{G}}$, $V^{\circ}_{\mathcal{G}}$, and $V^{!}_{\mathcal{G}} \cup V^{?}_{\mathcal{G}}$ are respecitely called *atomic*, *unit*, *jump*, and *modalic* vertices. An *RB-cograph* is an RGB-cograph $\mathcal{G}$ with $V_{\mathcal{G}} = V^{\bullet}_{\mathcal{G}}$.

The first condition in the definition says that if a vertex has an outgoing $\rightsquigarrow$-edge then it has to be modalic. The other conditions can be interpreted as follows: modalic vertices are grouped in sets containing a unique vertex in $V^{!}_{\mathcal{G}}$, representing instances of the $\mathsf{s}!p$-rule in the sequent calculus and box borders in the proof net syntax, while the jumps vertices are associated to either a pair of atomic vertices or a unit vertex[3].

In drawing an RGB-cograph we use bold (blue) edges v—w when $v \neq w$ and $v \curlyvee w$. Moreover, we allow us to omit to represent edges which can be deduced by $\curlyvee$ transitivity. For example, we draw u—v—w omitting the edge u—w.

Definition 10. An *æ-path* in an RGB-cograph $\mathcal{G}$ is an elementary path $x_0, x_1, \ldots, x_n$ in the graph $\langle V, \frown \cup \rightsquigarrow \cup \curlyvee \rangle$ whose edges are alternating in $\curlyvee$ and in $\rightsquigarrow \cup \frown$. A *chord* in an æ-path is an edge $x_i \frown x_j$ or $x_i \rightsquigarrow x_j$ for $i, j \in \{0, \ldots, n\}$ and $i+2 \leqslant j$. A *chordless æ-path* is an æ-path without chord. An *æ-cycle* is an æ-path such that $x_0 = x_n$. An RGB-cograph $\mathcal{G}$ is *æ-connected* if any two vertices are connected by a chordless æ-path, and $\mathcal{G}$ is *æ-acyclic* if it contains no chordless æ-cycle.

Definition 11. Let consider the following condition for an RGB-cograph $\mathcal{G}$:

1. $V_{\mathcal{G}} \neq \varnothing$ and $\mathcal{G}$ is æ-connected and æ-acyclic;
2. for every vertex $v \in V^{!} \cup V^{?}$ there is a vertex $w \in V^{\bullet} \cup V^{1} \cup V^{\circ}$ with $v \rightsquigarrow w$;
3. if $w \overset{\mathcal{G}}{\rightsquigarrow} v$ and $v \curlyvee v'$, then there is $w' \curlyvee w$ such that $w' \overset{\mathcal{G}}{\rightsquigarrow} v'$; and

We say that an RGB-cograph $\mathcal{G}$ is MELL-*correct* if it satisfies conditions 1, 2 and 3. It is is MLL-*correct* ($\mathsf{MLL_u}$-*correct*) if it satisfies conditions 1 and $V = V^{\bullet}$ (respectively $V = V^{\bullet} \cup V^{\circ}$).

Theorem 12. *Let $\mathcal{G}$ be a RGB-cograph with $\mathcal{G}_{[\![]\!]} = [\![F]\!]$ and $\mathsf{X} \in \{\mathsf{MLL}, \mathsf{MLL_u}, \mathsf{MELL}\}$. Then*

$$\mathcal{G} \text{ is the translation of a } \mathsf{X}^{\mathsf{LL}} \text{ proof of } F \iff \mathcal{G} \text{ is } \mathsf{X}\text{-correct}$$

Proof. The result for for $\mathsf{X} = \mathsf{MLL}$ is given in [27]. In fact, a MLL-correct RGB-cograph is an æ-connected æ-acyclic RB-cograph. In this proof, each $\curlyvee$-class $\{a, \bar{a}\}$ is associated to an ax-rule with conclusion $a, \bar{a}$. This result can be extended for $\mathsf{X} = \mathsf{MLL_u}$ by observing that each $\curlyvee$-classes corresponds to an application of 1_j-rule or ax_j-rule. The result for $\mathsf{X} = \mathsf{MELL}$ immediately follows from the one for K-correct RGB-cograph in [4] by applying the same argument for $\mathsf{MLL_u}$ in order to accommodate 1_j- and ax_j-rules. The full proof can be found in Appendix A □

5 Skew fibrations

Definition 13. Let $\mathcal{G}$ and $\mathcal{H}$ be mixed graphs. A *skew fibration* $f\colon \mathcal{G} \to \mathcal{H}$ is a function from $V_{\mathcal{G}}$ to $V_{\mathcal{H}}$ that preserves $\frown$ and $\rightsquigarrow$ (that is if $vR_{\mathcal{G}}w$ then $f(v) R_{\mathcal{H}} f(w)$ for $R \in \{\frown, \rightsquigarrow\}$), and f has the *skew lifting* property, i.e.,

$$\text{if } v \in V_{\mathcal{G}},\ w \in V_{\mathcal{H}},\ R \in \{\frown, \rightsquigarrow\} \text{ and } w R_{\mathcal{H}} f(v) \text{ , then } u R_{\mathcal{G}} v \text{ and } w \overset{\mathcal{H}}{\not\frown} f(u) \text{ and } w \overset{\mathcal{H}}{\not\rightsquigarrow} f(u) \text{ for a } u \in V_{\mathcal{G}}. \tag{5}$$

A skew fibration $f\colon \mathcal{G} \to \mathcal{H}$ is *modalic* if whenever $u \overset{\mathcal{G}}{\smile} v$ and $f(u) \overset{\mathcal{H}}{\rightsquigarrow} f(v)$, then there is a $w \in V_{\mathcal{G}}$ such that $w \overset{\mathcal{G}}{\rightsquigarrow} v$ and $f(u) = f(w)$, or $u \overset{\mathcal{G}}{\rightsquigarrow} w$ and $f(v) = f(w)$. A skew fibration is a *linear fibration* if modalic and it satisfies the following additional conditions:

[3] For readers familiar with proof nets syntax, $(V^{!} \cup V^{?})$-classes encodes boxes borders, $V^{!}$-vertices their principal ports, and jumps vertices are the placeholder for jumps and atomic vertices represent propositional variables.

1. if $l(v) = \circ$ then $l(f(v)) \in \{\bot, ?\}$;
2. if $\left|f^{-1}(v)\right| = 0$ then there is a unique $w \in V_{\mathcal{G}}$ such that $l(w) = \circ$, $l(f(w)) = ?$ and $f(w) \overset{\mathcal{H}}{\rightharpoonup} v$;
3. if $|f^{-1}(v)| = n > 1$ and $l(v) = ?$, then $\left|f^{-1}(w)\right| \geqslant n$ for all w such that $v \rightharpoonup w$, otherwise $f^{-1}(v) = \{v_1, \ldots, v_n\}$ with $n > 1$ and there are $\{w_1, \ldots, w_n\}$ such that $w_i \neq w_j$, $l(w_i) = ?$ and $f(w_i) \rightharpoonup f(v_i)$ for all $i, j \in \{1, \ldots, n\}$.

The additional conditions for linear fibration have the following interpretation: 1 associates to each jump a $\bot$ or the $?$ of a formula $?A$ introduced by a $\mathsf{w}_?$; in fact, condition 2 assures for every vertex v in $\mathcal{H}$ which has no pre-image, belongs to a subformula $?A$ which has been introduced by a $\mathsf{w}_?$. Condition 3 assures that if a vertex is image of multiple vertices, then there is a formula $?A$ such that this vertex is either the one corresponding to the $?$ or is a vertex of $[\![A]\!]$. These conditions allow us to restrict on the formulas on the form $?A$ the well-known correspondence between contractions-weakening derivations and skew fibrations [16, 28, 4, 3, 24].

Proposition 14. *If Γ and Γ' are sequents, then $f : [\![\Gamma']\!] \to [\![\Gamma]\!]$ is a linear weak fibration iff $\Gamma' \overset{\mathsf{w}_?, \mathsf{c}_?, \equiv}{\longmapsto} \Gamma$.*

In order to capture also $\mathsf{der}_?$ and $\mathsf{dig}_?$ rules application, we recall the following definition from [4].

Definition 15. We say that two vertices v and w in a relation web $\mathcal{G}$ are *clones* if for all u with $u \neq v$ and $u \neq w$ we have uRv iff uRw for all $R \in \{\frown, \rightharpoonup, \leftharpoondown, \smile\}$. If $v = w$ then they are trivially clones. A *?-map* is a mapping $f : \mathcal{G} \to \mathcal{H}$ where $\mathcal{G}$ and $\mathcal{H}$ are modalic and properly labeled relation webs, such that the following conditions are fulfilled:

- if $v \neq w$ and $f(v) = f(w)$, then v and w are clones in $\mathcal{G}$, $v \overset{\mathcal{G}}{\rightharpoonup} w$ and $l(f(v)) = l(f(w)) = ?$;
- if $f(v) \neq f(w)$ then $vR_{\mathcal{G}}w$ implies $f(v)R_{\mathcal{H}}f(w)$ for any $R \in \{\frown, \rightharpoonup, \leftharpoondown, \smile\}$;
- if $v \in V_{\mathcal{H}}$ is not in the image of f then $l(v) = ?$ and there is a $w \in V_{\mathcal{H}}$ with $v \rightharpoonup w$.

Analogously to the result in [4] for $\{4^\downarrow, \mathsf{t}^\downarrow\}$-maps, we have the following

Proposition 16. *Let Γ and Γ' be sequents. Then $\Gamma' \overset{\mathsf{der}_?^\downarrow, \mathsf{dig}_?^\downarrow}{\longmapsto} \Gamma$ iff $f : [\![\Gamma']\!] \to [\![\Gamma]\!]$ is a ?-map.*

We conclude this section we define a MELL-*fibration* $f : \mathcal{G} \to \mathcal{H}$ as the composition $f = f'' \circ f'$ a linear week fibration f' and a ?-map f''. As consequence of Propositions 14 and 16 we have the following

Theorem 17. *Let Γ and Γ' be sequents, then $\Gamma' \overset{\mathsf{c}_?^\downarrow, \mathsf{w}_?^\downarrow, \mathsf{der}_?^\downarrow, \mathsf{dig}_?^\downarrow}{\longmapsto} \Gamma$ iff $f : [\![\Gamma']\!] \to [\![\Gamma]\!]$ for an MELL-fibration f.*

6 Combinatorial Proofs

Definition 18. A map $f : \mathcal{G} \to \mathcal{F}$ from an RGB-cograph $\mathcal{G}$ to a modalic and properly labeled relation web $\mathcal{F}$ is *allegiant* if the following conditions are satisfied:

- if $v, w \in V_{\mathcal{G}}^\bullet$ and $v \overset{\mathcal{G}}{\curlyvee} w$ then $f(v)$ and $f(w)$ are labeled by dual atoms in $\mathcal{A}$;
- if $v \in V_{\mathcal{G}}^{\mathsf{1}}$ then $l(f(v)) = 1$; if $v \in V_{\mathcal{G}}^{!}$ then $l(f(v)) = !$; if $v \in V_{\mathcal{G}}^{?}$ then $l(f(v)) = ?$.

Definition 19. For $\mathsf{X} \in \{\mathsf{MLL}, \mathsf{MLL_u}, \mathsf{MELL}\}$, an X-*combinatorial proof* of a sequent Γ is an MELL-fibration $f : \mathcal{G} \to [\![\Gamma]\!]$ from an X-correct RGB-cograph $\mathcal{G}$ to the relation web of Γ.

Theorem 20. *If F is a formula and $\mathsf{X} \in \{\mathsf{MLL}, \mathsf{MLL_u}, \mathsf{MELL}\}$, then*

$$\overset{\mathsf{X}}{\longmapsto} F \iff \textit{there is a } \mathsf{X}\textit{-combinatorial proof } f : \mathcal{G} \to [\![F]\!]$$

Proof. By Proposition 1 and Theorem 3, if $\overset{\mathsf{X}}{\vdash} F$ then there is a formula F' such that $\overset{\mathsf{MELL}^{\mathsf{LL}}}{\vdash} F' \overset{\mathsf{MELL}^{\downarrow}}{\vdash} F$. We conclude by Theorems 12 and 17. □

Proposition 21. *If π and π' are two derivation in* $\mathsf{MELL}^{\mathsf{j}}$*, then $\pi \simeq_{\mathsf{J}} \pi'$ iff they are represented by the same* MELL*-combinatorial proof.*

Proof. After Remark 2, we assume π and π' to be jump-erasing free. Since Theorem 3 makes use only of rules preserving $\simeq_{\mathsf{J}}$, the rules in $\mathsf{MELL}^{\mathsf{LL}}$ permute with rules in $\mathsf{MELL}^{\downarrow}$. The RGB-cograph captures rules permutations in $\mathsf{MELL}^{\mathsf{LL}}$, and the skew fibration captures permutations in $\mathsf{MELL}^{\downarrow}$. □

To check if an RGB-cograph is X-correct and if a graph homomorphism is a MELL-fibration requires polynomial time in the size of the graphs. This gives us the following results.

Theorem 22. MELL*-combinatorial proofs are a sound and complete proof system for* MELL.

7 Handsome Proof Nets for MELL

The construction defined in the previous sections can be interpreted as an extension of both Retoré's [27] and Hughes' [16] syntaxes. However, combinatorial proofs represent cut-free proofs as *unfolded* Retoré's proof nets [26] do. We extend the combinatorial proof syntax to *handsome proof nets* in order to represent proofs with cuts. Our construction differs from [29], where the syntax of the absence of modalities allows more flexible representation of combinatorial proofs. In Appendix B, we give a normalization procedure to associate a combinatorial proof to any handsome proof net. Each step of this normalization procedure corresponds to a cut-elimination step in the sequent calculus $\mathsf{MELL}^{\mathsf{j}}$. The termination of the normalization follows MELL cut-elimination theorem [9]. Moreover, every cut-elimination step is deterministic because of fixed jump, hence the procedure is locally confluent.

Definition 23. An *(exponentially) handsome proof net* (or HPN for short) f is defined as follows:

- $f\colon \mathscr{G}_{\Gamma'} \to [\![\Gamma]\!]$ is a MELL-combinatorial proof;
- $f\colon \mathscr{G}_{\Gamma',A'} \to [\![\Gamma,A]\!]$ and $f\colon \mathscr{G}_{\Delta',\bar{A}'} \to [\![\Delta,\bar{A}]\!]$ are two HPN, then $f_1 \circ^{C}_{\mathsf{cut}} f_2 = f\colon \mathscr{G}_{\Gamma',\Delta'} \to [\![\Gamma,\Delta]\!]$ is an HPN.

In drawing such objects, we use bold (blue) edges v—w in order to connect each vertex $x \in [\![C]\!]$ with $l(x) = a$ with the corresponding vertex $\bar{x} \in [\![\bar{C}]\!]$ with label $\bar{a}$. For a graphical example refer to Figure 1.

8 Conclusions

In this paper we extend Retoré's cographic syntax for multiplicative proof nets [27] in order to include units and exponentials, using the results in [4] on combinatorial proofs for modal logic.

Aware of the limits designing a syntax with a polynomial correcness criterion [14], we restrain the notion of proof equicalence captured by the syntax. Our system allows rules permutations in Figure 4, provided that jump assignations are not changed. As a consequence, this proof equivalence can be checked in polynomial time. This notion of proof equivalence matches the one of "nouvelle syntaxe" proof net [25] with jumps, where $\mathsf{w}_?$ and $\mathsf{c}_?$ form a monad and their gates can be freely move inside and outside boxes.

The syntax presented in this paper is a coherence semantics for the system $\mathsf{MELL}^{\mathsf{j}}$ in the sense of [23], where we use of relation web instead of graphs. Hence, it can be further employed to explore the geometry of interaction [10] of MELL, similarly to what done for MLL in [8] using the original Retoré's syntax.

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A Proof ot Theorem 12

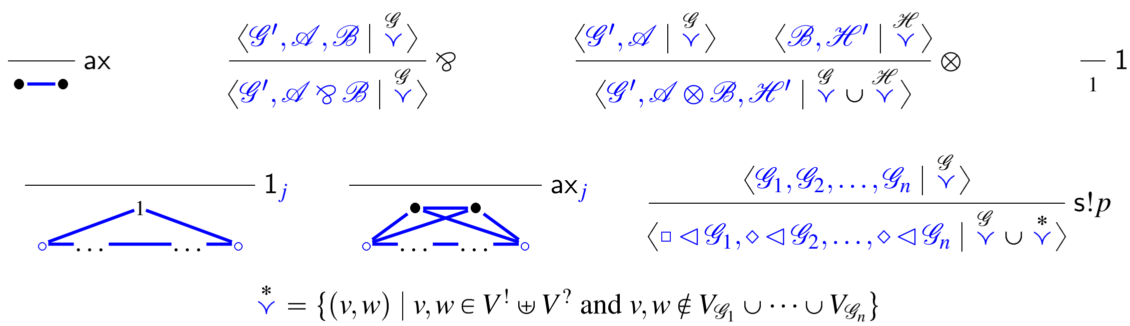

Figure 6: Translating $\mathsf{MELL}^{\mathsf{LL}}$ sequent proofs into RGB-cographs

Theorem 24. *Let $\mathscr{G}$ be a RGB-cograph with $\mathscr{G}_{[\![]\!]} = [\![F]\!]$ and $\mathsf{X} \in \{\mathsf{MLL}, \mathsf{MLL_u}, \mathsf{MELL}\}$. Then*

$$\mathscr{G} \text{ is the translation of a } \mathsf{X}^{\mathsf{LL}} \text{ proof of } F \iff \mathscr{G} \text{ is } \mathsf{X}\text{-correct}$$

Proof. $\mathsf{X} = \mathsf{MLL}$: in [27] is given a procedure associating to each MLL-correct RGB-cograph (i.e. a æ-connected æ-acyclic RB-cograph) a derivation in MLL. In particular, an $\curlyvee$-class $\{a, \bar{a}\}$ is associated to an ax-rule with conclusion $a, \bar{a}$;

$\mathsf{X} = \mathsf{MLL_u}$: For the left-to-right direction, observe that any vertex graph with a single $\curlyvee$-class is $\mathsf{MLL_u}$-correct. Furthermore, all rules in Figure 6 corresponds to rules in $\mathsf{MLL_u^{LL}}$ and preserve Condition 1.

To prove the right-to-left direction, we can use the sequentialization result for MLL on RB-cographs [27]. It suffices to extend this result by taking into account the cases when a $\curlyvee$-class is of the form $\{1, \circ, \dots, \circ\}$ or $\{a, \bar{a}, \circ, \dots, \circ\}$, and associate to it a 1_j-rule or respectively ax_j-rule.

$\mathsf{X} = \mathsf{MELL}$: The result follows the proof of K-correct RGB-cograph in [4] together with the argument of the previous point. The left-to-right direction, is similar to the one the previous case: all constructions given in Figure 6 corresponding to MELL rules preserve Conditions 1, 2 and 3.

For the right-to-left direction, we use again the MLL sequentialization result for RB-cographs [27] as extended in the previous case.

For this we will define for an RGB-cograph $\mathscr{G}$ an RB-cograph $\partial(\mathscr{G})$ that is æ-connected and æ-acyclic if and only if $\mathscr{G}$ is. To ease the notation, we give each vertex a unique label, such that $\bullet$-vertices that are linked are labeled by the atoms in the equivalence class, and $!$- and $?$-vertices are labeled by natural numbers, and we identify vertices with their labels. Now, $\partial(\mathscr{G})$ is obtained as follows. We define a vertex set $V^* = \{v', \bar{v}' \mid v \in V^!_{\mathscr{G}} \uplus V^?_{\mathscr{G}}\}$ and let $V_{\partial(\mathscr{G})} = V^\bullet_{\mathscr{G}} \uplus V^\circ_{\mathscr{G}} \uplus V^*$, i.e., we take the atomic vertices of $\mathscr{G}$, and each modalic vertex is replaced by a dual pair of atomic vertices, that are linked by $\overset{\partial(\mathscr{G})}{\curlyvee}$, and on vertices in $V^\bullet_{\mathscr{G}} \cup V^\circ_{\mathscr{G}}$, the relation $\overset{\partial(\mathscr{G})}{\curlyvee}$ is the same as in $\overset{\mathscr{G}}{\curlyvee}$. In order to define $\overset{\partial(\mathscr{G})}{\frown}$, we need an auxiliary relation:

$$x \overset{\mathscr{G}}{\frown\!\!\frown} y \quad \iff \quad x \overset{\mathscr{G}}{\frown} y \text{ and there is no } v \in V^!_{\mathscr{G}} \uplus V^?_{\mathscr{G}} \text{ with } x \overset{\mathscr{G}}{\frown} v \overset{\mathscr{G}}{\curvearrowright} y \text{ or } y \overset{\mathscr{G}}{\frown} v \overset{\mathscr{G}}{\curvearrowright} x$$

Now, we let $x \overset{\partial(\mathscr{G})}{\frown} y$ iff one of the following cases holds:

- $x, y \in V^{\bullet}_{\mathscr{G}}$ and $x \overset{\mathscr{G}}{\frown} y$;
- $x \in V^{\bullet}_{\mathscr{G}}$ and $y = w'$ for some $w \in V^{!}_{\mathscr{G}} \uplus V^{?}_{\mathscr{G}}$ with $x \overset{\mathscr{G}}{\frown} w$;
- $x = v'$ and $y = w'$ for some $v, w \in V^{!}_{\mathscr{G}} \uplus V^{?}_{\mathscr{G}}$ with $v \overset{\mathscr{G}}{\frown} w$;
- $x = \vec{v}'$ for some $v \in V^{!}_{\mathscr{G}} \uplus V^{?}_{\mathscr{G}}$ and $y \in V^{\bullet}_{\mathscr{G}} \cup V^{\circ}_{\mathscr{G}}$ with $v \overset{\mathscr{G}}{\rightharpoonup} y$;
- $x = \vec{v}'$ and $y = w'$ for some $v, w \in V^{!}_{\mathscr{G}} \uplus V^{?}_{\mathscr{G}}$ with $v \overset{\mathscr{G}}{\rightharpoonup} w$;
- $x = \vec{v}'$ for some $v \in V^{!}_{\mathscr{G}} \uplus V^{?}_{\mathscr{G}}$ and $y \in V^{\bullet}_{\mathscr{G}} \cup V^{\circ}_{\mathscr{G}}$ and there is a $u \in V^{!}_{\mathscr{G}} \uplus V^{?}_{\mathscr{G}}$ with $v \overset{\mathscr{G}}{\curlyvee} u \overset{\mathscr{G}}{\rightharpoonup} y$;
- $x = \vec{v}'$ and $y = w'$ for some $v, w \in V^{!}_{\mathscr{G}} \uplus V^{?}_{\mathscr{G}}$ and there is a $u \in V^{!}_{\mathscr{G}} \uplus V^{?}_{\mathscr{G}}$ with $v \overset{\mathscr{G}}{\curlyvee} u \overset{\mathscr{G}}{\rightharpoonup} w$;

The intuition behind this construction can be explained using Theorem 5. Following [12], we use the term BV-*formula* for an expression built from the atoms and $\circ$ using the binary operations $⅋$, $\otimes$, and $\lhd$, and it has been shown in [12] that the BV-formulas, modulo associativity and commutativity of $⅋$ and $\otimes$, and associativity of $\lhd$, are in one-to-one correspondence with the relation webs, via (2) and Proposition 8. We write $\mathsf{fm}(\mathscr{G})$ for a corresponding formula expression for $\mathscr{G}$, and we write $\partial(\mathsf{fm}(\mathscr{G}))$ for $\mathsf{fm}(\partial(\mathscr{G}))$. Now let $v_1, \ldots, v_n \in V^{!}_{\mathscr{G}} \uplus V^{?}_{\mathscr{G}}$ form an $\overset{\mathscr{G}}{\curlyvee}$-equivalence class. This means that $\mathsf{fm}(\mathscr{G})$ is of shape $F\{v_1 \lhd B_1\} \cdots \{v_n \lhd B_n\}$ for some n-ary context $F\{\ \} \cdots \{\ \}$ (because $\mathscr{G}$ is modalic). We can reformulate the translation above as follows:

$$\partial(F\{v_1 \lhd B_1\} \cdots \{v_n \lhd B_n\}) = (\vec{v}'_1 \otimes \cdots \otimes \vec{v}'_n \otimes \partial(B_1 ⅋ \cdots ⅋ B_n)) ⅋ \partial(F\{v'_1\} \cdots \{v'_n\}) \qquad (6)$$

We use (6) to construct $\partial(\mathscr{G})$ from $\mathscr{G}$ inductively on $\left|V^{!}_{\mathscr{G}} \uplus V^{?}_{\mathscr{G}}\right|$ and show that $\partial(\mathscr{G})$ is an RB-cograph iff $\mathscr{G}$ is an RGB-cograph, and that $\partial(\mathscr{G})$ is æ-connected and æ-acyclic iff $\mathscr{G}$ is.

For this, observe that, a priori, moving a B_i out from the context could create or destroy æ-paths. However, we only claim that æ-connectedness and æ-acyclicity are preserved, i.e., if the original RGB-cograph is correct, then so is the one constructed via 6, and vice versa. So, by way of contradiction, assume the one in the right-hand side sequent of 6 contains a chordless æ-cycle. This cycle cannot contain atoms from both $\vec{v}'_1 \otimes \cdots \otimes \vec{v}'_n \otimes \mathsf{fm}(B_1 ⅋ \cdots ⅋ B_n)$ and $\mathsf{fm}(F\{v'_1\} \cdots \{v'_n\})$ because they are connected by a $⅋$. Hence, the cycle cannot contain any v'_i or $\vec{v}'_i$. This means that the cycle is fully contained inside the context $F\{\ \} \cdots \{\ \}$ or inside one of the B_i. Therefore the cycle must already be present in the original RGB-cograph. Contradiction. Now pick any two vertices x' and y' in the right-hand side sequent of 6. We show that there is a chordless æ-path between them. Let x and y be the corresponding vertices in the original RGB-cograph (if x' or y' are one of the v'_i or $\vec{v}'_i$, take the corresponding v_i). By the assumption there is a chordless æ-path between x and y. We can recover this path in the right-hand side sequent of 6. If the original path passes through a v_i, we can in the new graph pass through the new edge v'_i—$\vec{v}'_i$. The converse is proved similarly.

Figure 7 shows two examples. We can now piggyback on Retoré's proof [27] of sequentialization for RB-cographs, in order to produce an $\mathsf{MELL}^{\mathsf{LL}}$ sequent proof for $\mathsf{fm}(\mathscr{G})$. Since $\partial(\mathscr{G})$ is æ-connected and æ-acyclic RB-cograph, there is a splitting tensor in $\mathsf{fm}(\partial(\mathscr{G}))$ (we can remove root $⅋$ via the $\vee$-rule). If this splitting tensor is also present in $\mathsf{fm}(\mathscr{G})$, we can directly apply the $\wedge$ rule and proceed by induction hypothesis. If it is not present in $\mathsf{fm}(\mathscr{G})$ then it must be of shape $\vec{v}'_1 \otimes \cdots \otimes \vec{v}'_n \otimes \partial(B_1 ⅋ \cdots ⅋ B_n)$ and be introduced by the translation (6). Since $\partial(\mathscr{G})$ is æ-connected, we can without loss of generality assume the the context consists only of $v'_1, \ldots, v'_n$. Otherwise our tensor would not be splitting. Hence, we have

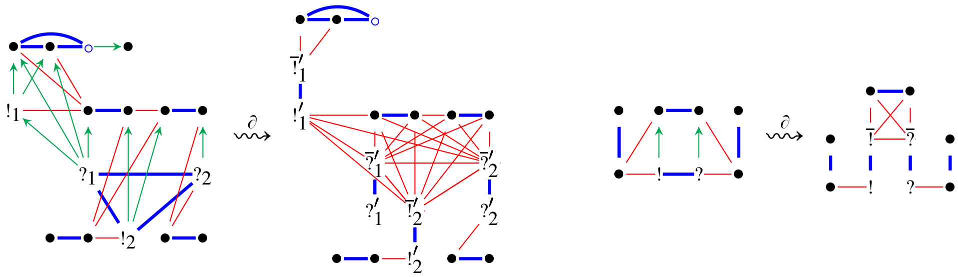

Figure 7: The RGB-cographs for $F_1 = \bar{d} \parr (d \otimes !(\bar{b} \otimes c) \parr \bar{e} \parr (e \otimes ?\bar{c}) \parr ?(b \otimes !(a \parr \bar{a}))) \parr ?f$ and $F_2 = b \parr (\bar{b} \otimes !a) \parr (?\bar{a} \otimes c) \parr \bar{c}$, and the corresponding RB-cographs $\partial(F_1)$ and $\partial(F_2)$.

$$\otimes \dfrac{\otimes \dfrac{\mathsf{ax}\,\dfrac{}{v'_1, \bar{v}'_1} \quad \cdots \quad \mathsf{ax}\,\dfrac{}{v'_n, \bar{v}'_n}}{v'_1, \dots, v'_n, \bar{v}'_1 \otimes \cdots \otimes \bar{v}'_n} \qquad \partial(B_1 \parr \cdots \parr B_n)}{v'_1, \dots, v'_n, \bar{v}'_1 \otimes \cdots \otimes \bar{v}'_n \otimes \partial(B_1 \parr \cdots \parr B_n)} \tag{7}$$

whose conclusion is $\partial((v_1 \lhd B_1) \parr \cdots \parr (v_n \lhd B_n))$. Thus, we can apply the $\mathsf{s}!p$-rule and we can proceed by induction hypothesis.

Moreover, if $V^{\circ}_{\mathcal{G}} \neq \emptyset$, then accordingly with the previous translation $\mathsf{fm}(\mathcal{G}) = F\{v_1 \lhd A_1\} \cdots \{v_n \lhd A_n\}$ and $\partial(\mathsf{fm}(\mathcal{G})) = F\{v_1\} \cdots \{v_n\}$. Then derivation corresponding to $\partial(\mathcal{G})$ has an axiom (with jumps) with conclusions $w, \bar{w}, \circ_1, \dots, \circ_n$ where $\{w, \bar{w}, \circ_1, \dots, \circ_n\}$ is a $\curlyvee$-equivalence class which corresponds to the ax_j with conclusion $w, \bar{w}, \circ_1, \dots, ?\circ_1$.

□

B Handsome (Proof Nets) Normalization

We introduce the following notation:

- *relation web restriction*: if $\mathcal{G}$ is a relation web and $X \subset V_{\mathcal{G}}$, then we define the relation web $\mathcal{G}|_X = \langle X, \frown \cap (X \times X), \rightharpoonup \cap (X \times X) \rangle$.
- *restricted fibration*: if $X \subset V_{\mathcal{G}}$, and $f : \mathcal{H} \to \mathcal{G}$ a skew fibration, then we define the skew fibration $f|_X : \mathcal{H}|_{f^{-1}(X)} \to \mathcal{G}|_X$, that a skew fibration from the vertices in $\mathcal{H}$ which have an image in $\mathcal{G}|_X$ which behaves like f;
- *identity fibration*: if Γ is a sequent, we define the identity MELL-fibration $1_\Gamma : [\![\Gamma]\!] \to [\![\Gamma]\!]$;
- *digging fibration*: if Γ is a sequent, we define the MELL-fibration $\mathsf{dig}(??\Gamma) : [\![??\Gamma]\!] \to [\![?\Gamma]\!]$ such that $\mathsf{dig}(??\Gamma)|_\Gamma = 1_\Gamma$;
- *jump fibration*: if $\circ \in V^{\circ}$ and $F \in \{\bot, ?A\}_{A \in \mathcal{L}}$, we define the MELL-fibration $0_A : [\![\circ]\!] \to [\![F]\!]$;
- *class merging*: if $\mathcal{G} = \langle V, \frown, \rightharpoonup, \curlyvee \rangle$ is an RGB-cograph, ρ_1, ρ_2 two $\curlyvee$-equivalence classs of $\mathcal{G}$ with $x_1 \in \rho_1$ and $x_2 \in \rho_2$ such that there is no $y \in V$ such that $x_i \frown y$, $x_i \rightharpoonup y$ or $y \rightharpoonup x_i$ for $i \in \{1,2\}$, then we define the RGB-cograph $\mathcal{G}[\rho \overset{x,x'}{\curlyvee} \rho'] = \langle V \setminus \{x_1, x_2\}, \frown, \rightharpoonup, \curlyvee' \rangle$ where $\curlyvee'$ is obtained by merging the two classes ρ and ρ' and removing the vertices x_1 and x_2

$$\curlyvee' = (\curlyvee \cup \{(y_1, y_2) \mid y_1 \in \rho_2, y_2 \in \rho_2\}) \setminus \{(x, y) \mid x \in \{x_1, x_2\}\}$$

- *class elimination*: if $\mathscr{G} = \langle V, \frown, \curvearrowright, \curlyvee \rangle$ is an RGB-cograph, $\rho \subset V^!_{\mathscr{G}} \cup V^?_{\mathscr{G}}$ is a $\curlyvee$-equivalence class of $\mathscr{G}$, then we define the RGB-cograph $(\mathscr{G} \backslash \rho)$ given by the relation web $\langle V, \frown, \curvearrowright \rangle|_{V \backslash \rho}$ and the equivalence relation $\curlyvee \backslash \{(x,y) | x, y \in \rho\}$.
- *jump replacing*: if $\mathscr{G} = \langle V, \frown, \curvearrowright, \curlyvee \rangle$ is an RGB-cograph, $x \in V^\circ$ and ρ is the unique equivalence class containing x, we define $\mathscr{G}[\mathscr{G}/\circ] = \langle V', \frown, \curvearrowright, \curlyvee' \rangle$ with
  - $\curlyvee' = (\curlyvee \backslash \{xy \in \curlyvee \mid y \in V\}) \cup \{\circ_i y \mid x \curlyvee y, i \in \{1, \dots, n\}\}$;
  - $V' = V^\bullet_{\mathscr{G}} \cup V^!_{\mathscr{G}} \cup V^?_{\mathscr{G}} \cup (V^\circ_{\mathscr{G}} \backslash \{x\} \cup \{\circ_1, \dots, \circ_n\})$.
- *RGB soft-promotion*: if $\mathscr{G} = \langle V, \frown, \curvearrowright, \curlyvee \rangle$ is an RGB-cograph such that $\mathscr{G}_{[\![]\!]} = [\![A, B_1, \dots, B_n]\!]$, we define the RGB-cograph $\mathsf{s}!p[\mathscr{G}]_{!A}$ with
  - $V_{\mathsf{s}!p[\mathscr{G}]_{!A}} = V^\bullet_{\mathscr{G}} \cup (V^!_{\mathscr{G}} \cup \{x\}) \cup (V^?_{\mathscr{G}} \cup \{y_1, \dots, y_n\}) \cup V^\circ_{\mathscr{G}}$;
  - $\overset{\mathsf{s}!p[\mathscr{G}]_{!A}}{\frown} = \overset{\mathscr{G}}{\frown}$;
  - $\overset{\mathsf{s}!p[\mathscr{G}]_{!A}}{\curvearrowright} = \overset{\mathscr{G}}{\curvearrowright} \cup (\{xa \mid a \in V_{[\![A]\!]}\} \cup \{y_i b_i \mid b_i \in V_{[\![B_i]\!]}\}_{i \in \{1,\dots,n\}})$;
  - $\overset{\mathsf{s}!p[\mathscr{G}]_{!A}}{\curlyvee} = \overset{\mathscr{G}}{\curlyvee} \cup \{uv \mid u, v \in \{x, y_1, \dots, y_n\}, u \neq v\}$.
- *class elimination* if $\mathscr{G}$ is a RGB-cograph and ρ a $\curlyvee$-equivalence class such that there are no $x \in V_{\mathscr{G}}$ with $x \frown y$ or $x \curvearrowright y$ for any $y \in \rho$, then we define the combinatorial proof
$$(f \backslash \rho) \colon (\mathscr{G} \backslash \rho) \to \mathscr{G}|_{f(V_{(\mathscr{G} \backslash \rho)})}$$
- *soft promotion* if $f = f \colon \mathscr{G}_{A', B'_1, \dots, B'_n} \to [\![A, B_1, \dots, B_n]\!]$ is a MELL-combinatorial proof, then we define the MELL-combinatorial proof
$$\mathsf{s}!p[f]_{!A} \colon \mathsf{s}!p[\mathscr{G}]_{!A'} \to [\![!A, ?B_1, \dots, wnB_n]\!]$$
such that $\mathsf{s}!p[f]_{!A}|_{V_{[\![\Gamma]\!]}} = f$.

Lemma 25 (Splitting). *If $f \colon \mathscr{G}_{\Gamma', A' \otimes B'} \to [\![\Gamma, A \otimes B]\!]$ is a combinatorial proof, then there are Γ'_1, Γ'_2 sequents such that $\Gamma'_1 \cup \Gamma'_2 = \Gamma'$, $f_1 \colon \mathscr{G}_{\Gamma'_1, A'} \to [\![\Gamma_1, A]\!]$ and $f_2 \colon \mathscr{G}_{\Gamma'_2, B'} \to [\![\Gamma, B]\!]$ combinatorial proofs such that $\mathscr{G}_{\Gamma', A' \otimes B'} = \langle \mathscr{G}_{\Gamma'_1}, \mathscr{A}' \otimes \mathscr{B}', \mathscr{G}_{\Gamma'_2} \mid \overset{\mathscr{G}}{\curlyvee} \cup \overset{\mathscr{H}}{\curlyvee} \rangle$ and*
$$f|_A = f_1|_A \qquad f|_{\Gamma_1} = f_1|_{\Gamma_1} \qquad f|_B = f_2|_B \qquad f|_{\Gamma_2} = f_2|_{\Gamma_2}$$

Proof. It follows the similar result for RB-cographs. It suffices to remark that MELL-fibrations preserve spitting sections. □

Definition 26 (Normalization). If $f_1 \colon \mathscr{G}_{C', \Gamma'} \to [\![C, \Gamma]\!]$ and $f_2 \colon \mathscr{G}_{\bar{C}', \Delta'} \to [\![\bar{C}, \Delta]\!]$ are two MELL combinatorial proofs, then we define a normalization step
$$f_1 \circ^C_{\mathsf{cut}} f_2 \rightsquigarrow f \colon \mathscr{G}_{\Gamma', \Delta'} \to [\![\Gamma, \Delta]\!]$$
associatin to the HPN $f_1 \circ^C_{\mathsf{cut}} f_2$ a MELL combinatorial proof. Each step is recursively defined as follows:

- if $C \in \{\bot, a\}$ with $a \in \mathscr{A}$ then f_1 and f_2 are bijective respectively on C and $\bar{C}$. Let $x_1 \in V_{\mathscr{G}_{C', \Gamma'}}$ and $\bar{x}_2 \in V_{\mathscr{G}_{\bar{C}', \Delta'}}$ be the vertices which image is respectively $[\![C]\!]$ and $[\![\bar{C}]\!]$ and ρ_1 and ρ_2 the corresponding $\curlyvee$-classes containing x_1 and x_2. Then
$$f_1 \circ^C_{\mathsf{cut}} f_2 = f \colon \mathscr{G}[\rho_1 \overset{x_1, x_2}{\curlyvee} \rho_2] \to [\![\Gamma, \Delta]\!]$$
where $f|_\Gamma = f_1|_\Gamma$ and $f|_\Delta = f_2|_\Delta$;

- if $C = A \otimes B$, by Lemma 25 there are two RGB-cographs $\mathcal{G}_{\Gamma'_1,A'}$ and $\mathcal{G}_{\Gamma'_2,B'}$ such that $\Gamma' = \Gamma'_1, \Gamma'_2$ and $\mathcal{G}_{\Gamma',A\otimes B} = \langle \mathcal{G}_{\Gamma'_1}, \mathscr{A} \otimes \mathscr{B}, \mathcal{G}_{\Gamma'_2} \mid \overset{\mathcal{G}}{\curlyvee} \cup \overset{\mathcal{H}}{\curlyvee} \rangle$. If $f_1^A = f_1|_{[\![\Gamma_1,A]\!]}$ and $f_1^B = f_1|_{[\![\Gamma_2,B]\!]}$ and then

$$f_1 \circ^C_{\mathsf{cut}} f_2 = f_1^A \circ^A_{\mathsf{cut}} f_2 \circ^B_{\mathsf{cut}} f_1^B$$

- if $C = !A$, let us denote by $!$ and $?$ the vertices in $[\![!A]\!]$ and $[\![?\bar{A}]\!]$ corresponding to the modality $!$ of A and $?$ of $\bar{A}$. We have the following cases:
  - $\mathsf{s}!p$ vs ax_j: if $f_2^{-1}(?) = \{\circ\}$ with x belonging to the $\curlyvee$-equivalence class ρ, and $f_1 \colon \mathcal{G}_1 \to [\![!A, ?B_1, \dots, ?B_n]\!]$, then

$$f_1 \circ^C_{\mathsf{cut}} f_2 = f : \mathcal{G}[\circ_1, \dots \circ_n / \circ] \to [\![\Delta, ?B_1, \dots, ?B_n]\!]$$

    with $f|_\Delta = f_2|_\Delta$ and $f|_{?B_i} = 0_{?B_i}$.
  - $\mathsf{s}!p$ vs $\mathsf{s}!p$: if $f_1|_C$ and $f_2|_{\bar{C}}$ are injective and $\rho_1^x, \rho_2^{\bar{x}}$ are the corresponding $\curlyvee$-equivalence classes in $\mathcal{G}_{\Gamma',C'}$ and $\mathcal{G}_{\Delta',\bar{C}'}$ of each $x \in \{x_1, \dot{,} x_n\} = V_{[\![C]\!]}$ and $\bar{x} \in \{\bar{x}_1, \dots, \bar{x}_n\} = V_{[\![\bar{C}]\!]}$, then

$$f_1 \circ^C_{\mathsf{cut}} f_2 = \mathcal{G}[\rho_1^{x_1} \overset{x_1,\bar{x}_1}{\curlyvee} \rho_2^{\bar{x}_1}] \dots [\rho_1^{x_n} \overset{x_n,\bar{x}_n}{\curlyvee} \rho_2^{\bar{x}_n}]$$

  - $\mathsf{s}!p$ vs $\mathsf{der}_?$: if $f_2^{-1}(?) = \varnothing$ and $f_2^{-1}(x) \neq \varnothing$ for all $x \in V_{[\![?\bar{A}]\!]}$ such that $? \rightsquigarrow x$, and if ρ is the $\curlyvee$-equivalence class containing the $!$, then

$$f_1 \circ^C_{\mathsf{cut}} f_2 = (f_2 \backslash \rho) \circ^A_{\mathsf{cut}} f_2|_{V_{\mathcal{G}} \setminus \{?\}}$$

  - $\mathsf{s}!p$ vs $\mathsf{dig}_?$: if $|f^{-1}(?)| = n > 1$ and $|f_2^{-1}(v)| = 1$ for all $v \in V_{[\![?\bar{A}]\!]}$, then

$$f_1 \circ^C_{\mathsf{cut}} f_2 = f_1 \circ^C_{\mathsf{cut}} (\mathsf{s}!p[f_2]_{!!A} \circ (1_{!!A} \parr \mathsf{dig}(??\Delta)))$$

  - $\mathsf{s}!p$ vs $\mathsf{c}_?$: if $|f_2^{-1}(?)| = n > 1$ and $|f_2^{-1}(v)| = n$ for at least one $v \in V_{[\![?\bar{A}]\!]}$, we define $f'_2 \colon \mathcal{G}_2 \to [\![\Delta]\!], ?\bar{A}_1, \dots ?\bar{A}_n$ such that $f'_2|_\Delta = f_2|_\Delta$ and $f'_2|_{?\bar{A}_1,\dots ?\bar{A}_n}$ is injective on the vertices $\{?_1, \dots, ?_n\}$ encoding the (principal) modalities of $?\bar{A}_1, \dots ?\bar{A}_n$. Then

$$f_1 \circ^C_{\mathsf{cut}} f_2 = \underbrace{f_1 \circ^C_{\mathsf{cut}} \cdots \circ^C_{\mathsf{cut}} (f_1}_{n \text{ copies}} \circ^C_{\mathsf{cut}} f'_2))$$

Theorem 27. *If* $f \colon \mathcal{G}_{\Gamma'} \to [\![\Gamma]\!]$ *is an* HPN*, then there is a well-defined* MELL *combinatorial proof* $\hat{f} \colon \mathcal{G}_{\Gamma'} \to [\![\Gamma]\!]$ *such that* $f \colon \mathcal{G}_{\Gamma'} \to [\![\Gamma]\!] \rightsquigarrow^* \hat{f} \colon \mathcal{G}_{\Gamma'} \to [\![\Gamma]\!]$.

Proof. The proof follows the cut-elimination result for the proof system $\mathsf{MELL}^{\mathsf{j}}$. This latter follows the cut-elimination theorem for MELL [9] and Proposition 1. □