A Deep Inference System for Differential Linear Logic

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Differential linear logic (DiLL) provides a fine analysis of resource consumption in cut-elimination and a logical basis for a theory of approximation via the notion of Taylor expansion. We investigate the subsystem of DiLL without promotion via the tools of deep inference in order to push cut elimination at an atomic level. We prove that in our system every provable formula admits a derivation in normal form. Moreover, we provide a procedure to normalize derivation.

1 Introduction

Girard [10] introduced linear logic (LL) as a refinement of intuitionistic and classical logics, built around cut elimination. In LL, a pair of dual modalities (the *exponentials* ! and ?) give a logical status to the operations of erasing and copying (sub-)proofs in the cut elimination procedure. The idea is that *linear* proofs (*i.e.* proofs without exponentials) use their hypotheses exactly once, whilst *exponential* proofs may use their hypotheses at will. In particular, the *promotion rule* makes a (sub-)proof available to be erased or copied an unbounded number of times, provided that its hypotheses are as well (it is a contextual rule). Via Curry–Howard correspondence between programs and proofs, LL gives a logical status to the operations of erasing and copying data in the evaluation process. Linear proofs correspond to programs which call their arguments exactly once, exponential proofs to programs which call their arguments at will. The study of LL contributed to unveil the logical nature of resource consumption.

The importance of being differential. A further tool for the analysis of resource consumption came from Ehrhard and Regnier's work on differential λ-calculus [4] and differential linear logic (DiLL, [6, 17]). Despite the fact that DiLL is inconsistent (every sequent $\vdash \Gamma$ can be proved), it has a cut-elimination theorem [17, 9] and internalizes notions from denotational models of LL into the syntax. In particular, DiLL_0 (the *promotion-free* fragment of DiLL, [6]) is a syntax corresponding to the semantic constructions defined by Ehrhard's finiteness spaces [2]. Finiteness spaces interpret linear proofs as linear functions on certain topological vector spaces, on which one can define an operation of derivative. Exponential proofs are interpreted as analytic functions, in the sense that they can be arbitrarily approximated by the semantic equivalent of a Taylor expansion [2, 3], which becomes available thanks to the presence of a derivative operator. In syntactic terms, these constructions take a interesting form: they correspond to "symmetrizing" the exponential modalities, *i.e.* in DiLL_0 the rules handling the dual exponential modalities ! and ? are perfectly symmetrical, although the logic is not self-dual.

Indeed, in LL, only the promotion rule introduces the ! modality, creating inputs that can be called an unbounded number of times. In DiLL_0 the rules handling the ! modality (!-dereliction !d, !-contraction !c, !-weakening !w) are the duals of the usual rules dealing with the ? modality (?-dereliction ?d, ?-contraction ?c, ?-weakening ?w). In particular, !-*dereliction* expresses in the syntax the semantic derivative: it releases inputs of type $!A$ that must be called exactly *once*, so that executing a program f on a "!-derelicted" input x (*i.e.* performing cut elimination on a proof f cut with a "!-derelicted" sub-proof x) amounts to compute the best linear approximation of f on x. This imposes non-deterministic choices: if

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in an evaluation the program f needs several copies of the input x (*i.e.* if the proof f uses several times the hypothesis $!A$), then there are different executions of f on x, depending on which sub-routine (*i.e.* hypothesis) of f is fed with the unique available copy of x. Thus we get a formal *sum*, where each term represents a possibility. The rules !-contraction and !-weakening put together a finite (possibly 0) number of copies of an input, so that it can be called a *bounded* number of times during execution.

What is also interesting is that LL promotion rule can be encoded in DiLL_0 through the notion of syntactic *Taylor expansion* [5, 7, 16, 18]: a proof of LL can be decomposed into a possibly infinite set of (promotion-free) proofs of DiLL_0. The idea, given a proof in LL with exactly one promotion rule $!\mathsf{p}$, is to replace $!\mathsf{p}$ (which makes the resource π available at will) with an infinite set of "differential" proofs of DiLL_0, each of them taking $n \in \mathbb{N}$ copies of π so as to make the resource π available exactly n times. The potential infinity of the promotion rule becomes an actual infinite via the Taylor expansion.

Nets vs. sequents. The symmetry of the rules handling the dual exponentials ! and ? in DiLL_0 is evident using Lafont's interaction nets [15] (a graphical representation of proofs similar to LL proof-nets), but not at all in the sequent calculus formulation of DiLL_0. Besides, interaction nets allows one to express DiLL_0 cut elimination with a sharper account than in sequent calculus, getting rid of (many!) commutative cuts. Not by chance, all papers dealing with DiLL_0 cut elimination use only interaction nets [6, 17, 9, 19].

The interaction net presentation of DiLL_0 has some flaws: its objects do not have an inductive tree-like structure and so it is not easy to handle them. Moreover, not all these objects correspond to a derivation in DiLL_0 sequent calculus, a global correctness criterion is required to identify them.

Our contribution. We define a proof system for DiLL_0 in the formalism of the *open deduction* [13] following the principles of *deep inference* [14, 1, 12]. Such a formalism, which allows rules to be applied deep in a context, provides a more flexible composition of derivations and can explicit the behavior of the cut-elimination process in DiLL_0 in a more fine-grained way. Besides, our deep inference system for DiLL_0 gathers good qualities of both sequent calculus and interaction nets formalisms: it restores the interaction net *symmetries* lost in the sequent calculus and its derivations keep a handy *inductive* tree-like (or better, sequence-like) structure as in the sequent calculus, without the need for a global correctness criterion like in interaction nets.

A first attempt in the direction of a deep inference system for DiLL_0 is in [8] where, however, the sum-rule is absent and, as consequence, it is not suitable to represent the dynamic behavior of DiLL_0.

To fully recover the expressiveness of this logic, we design our system to include a binary connective $+$ which represents the sum operation. The rules for $+$ (and for its unit 0) prevent the use of Guglielmi and Tubella's general result [21, 22] to show cut-elimination. However, we are able to define a normalization procedure by rule permutations which fully captures the dynamics of DiLL_0 cut-elimination.

Thanks to the symmetry of rules in our formalism, we are also able to reduce cut-rule to an *atomic* level. In particular, we can classify our rule permutations based on their behavior: some rule permutations correspond to multiplicative cut-elimination steps, other permutations correspond to "resource management" cut-elimination steps (involving the ? and ! rules), other permutations correspond to "slice management" operations (involving the propagation of $+$ and 0).

2 Differential Linear Logic

We present here the classical, propositional, one-sided sequent calculus for *differential linear logic* without promotion (DiLL_0). The formulas of DiLL_0 are exactly the same as in the multiplicative exponential

$$\frac{}{\vdash A,\overline{A}}\,\mathsf{ax} \qquad \frac{\vdash \Gamma,A \quad \vdash \overline{A},\Delta}{\vdash \Gamma,\Delta}\,\mathsf{cut} \qquad \frac{\vdash \Gamma,A,B,\Delta}{\vdash \Gamma,B,A,\Delta}\,\mathsf{exc} \qquad \frac{\vdash \Gamma,A \quad \vdash B,\Delta}{\vdash \Gamma,A\otimes B,\Delta}\,\otimes \qquad \frac{\vdash \Gamma,A,B}{\vdash \Gamma,A \parr B}\,\parr \qquad \frac{}{\vdash 1}\,1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma,\bot}\,\bot$$

$$\frac{\vdash \Gamma,A}{\vdash \Gamma,!A}\,!\mathsf{d} \qquad \frac{\vdash \Gamma,A}{\vdash \Gamma,?A}\,?\mathsf{d} \qquad \frac{\vdash \Gamma,!A \quad \vdash !A,\Delta}{\vdash \Gamma,!A,\Delta}\,!\mathsf{c} \qquad \frac{\vdash \Gamma,?A,?A}{\vdash \Gamma,?A}\,?\mathsf{c} \qquad \frac{}{\vdash !A}\,!\mathsf{w} \qquad \frac{\vdash \Gamma}{\vdash \Gamma,?A}\,?\mathsf{w} \qquad \frac{}{\vdash \Gamma}\,\mathsf{zero} \qquad \frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma}\,\mathsf{sum}$$

Figure 1: Sequent calculus rules for DiLL_0 (differential linear logic without promotion) [17].

fragment of linear logic (MELL). MELL *formulas* are defined by the grammar below, where $a,b,c,\dots$ range over a countably infinite set of propositional variables:

$$A,B ::= a \mid \overline{a} \mid 1 \mid \bot \mid A\otimes B \mid A \parr B \mid !A \mid ?A$$

Linear negation $\overline{(\cdot)}$ is defined through De Morgan laws so as to be involutive ($\overline{\overline{A}} = A$ for any A):

$$\overline{(a)} = \overline{a} \qquad \overline{(\overline{a})} = a \qquad \overline{A\otimes B} = \overline{A} \parr \overline{B} \qquad \overline{A \parr B} = \overline{A}\otimes\overline{B} \qquad \overline{1} = \bot \qquad \overline{\bot} = 1 \qquad \overline{!A} = ?\overline{A} \qquad \overline{?A} = !\overline{A}$$

Variables and their negations are *atomic*; $\otimes, \parr$ are *multiplicative connectives* and $1,\bot$ are their respective *units*; $!,?$ are *exponential modalities*. A MELL *sequent* is a finite sequence of MELL formulas $A_1,\dots,A_n$ (for any $n \geq 0$), and it is ranged over by Γ,Δ,Σ. Figure 1 gives the sequent calculus rules[1] for *differential linear logic* DiLL_0 (without *promotion*); the rules on the first line correspond to the *multiplicative linear logic* fragment MLL. From now on, the use of the exchange rule exc is left implicit.

We define $\equiv$ as the equivalence relation on the derivations of DiLL_0 generated by the relations in Figure 2. Roughly, the rule zero plays the role of annihilating element with respect to all the other rules but sum, for which it is a neutral element; whilst the rule sum commutes with any rule below it. Clearly, $\equiv$ preserves conclusions and can be oriented so as to define a terminating rewriting relation that pushes down the rules zero and sum in a derivation. As a consequence, every derivation in DiLL_0 can be rewritten in a *canonical form* (with the same conclusions).

Definition 2.1 (Canonical form, slice)**.** Let π be a derivation in DiLL_0:

1. π is a *slice* if it is in $\mathsf{DiLL}_0^- = \mathsf{DiLL}_0 \setminus \{\mathsf{sum},\mathsf{zero}\}$ (*i.e.* the rules zero and sum do not occur in π);
2. π is *canonical* or in *canonical form* if either it consists of a zero rule, or it is a slice, or if its last rule is sum with a canonical form as left premise and a slice as right premise.

A *canonical form* of π is any canonical derivation π' in DiLL_0 such that $\pi \equiv \pi'$.

Fact 2.2 (Canonicity)**.** *Any derivation in* DiLL_0 *has a canonical form (with same conclusions).*

Intuitively, considering only canonical derivations, slices are the "real proofs" in DiLL_0 (corresponding to simple nets in [6, 16, 17]), while the rules sum and zero are needed to define cut elimination in DiLL_0 (see below). The rule sum puts together slices with the same conclusions $\vdash\Gamma$, similarly to multiset union: it expresses the possibility of several "real proofs" of $\vdash\Gamma$. The rule zero then corresponds to the empty multiset of "real proofs" of $\vdash\Gamma$: it claims $\vdash\Gamma$ without a proof (it is reminiscent of daimon in ludics [11]). Because of the rule zero, any MELL sequent (possibly the empty one) is provable in DiLL_0.

A derivation in DiLL_0 is with *atomic axioms* if every instance of the rule ax introduces a MELL sequent of the form $\vdash a,\overline{a}$, where a is a propositional variable.

Proposition 2.3 (Atomic axioms)**.** *For every derivation π in* DiLL_0 *with conclusion $\vdash \Gamma$, there exists a derivation π' in* DiLL_0 *with conclusion $\vdash \Gamma$ and atomic axioms. If, moreover, π is canonical (resp. a slice) then π' is canonical (resp. a slice).*

Claim p. 3
Proof p. 11

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[1] Usually, in the literature on LL and DiLL, the rules ?w, ?d, ?c, !w, !d, !c are called respectively weakening, dereliction, contraction, co-weakening, co-dereliction and co-contraction. To avoid clashes with the usual terminology in deep inference (see Footnote 2), we call them ?-weakening, ?-dereliction, ?-contraction, !-weakening, !-dereliction and !-contraction, respectively.

$$\dfrac{\dfrac{}{\vdash\Gamma}\mathsf{zero}}{\vdash\Delta}\mathsf{r}_1 \equiv \dfrac{}{\vdash\Delta}\mathsf{zero} \qquad \dfrac{\dfrac{}{\vdash\Gamma}\mathsf{zero} \quad \vdash\Delta}{\vdash\Sigma}\mathsf{r}_2 \equiv \dfrac{}{\vdash\Sigma}\mathsf{zero} \equiv \dfrac{\vdash\Gamma \quad \dfrac{}{\vdash\Delta}\mathsf{zero}}{\vdash\Sigma}\mathsf{r}_2 \qquad \dfrac{\dfrac{}{\vdash\Gamma}\mathsf{zero} \quad \vdash\Gamma}{\vdash\Gamma}\mathsf{sum} \equiv \vdash\Gamma \equiv \dfrac{\vdash\Gamma \quad \dfrac{}{\vdash\Gamma}\mathsf{zero}}{\vdash\Gamma}\mathsf{sum} \qquad \dfrac{\dfrac{\vdash\Gamma \quad \vdash\Gamma}{\vdash\Gamma}\mathsf{sum}}{\vdash\Delta}\mathsf{r}_1 \equiv \dfrac{\dfrac{\vdash\Gamma}{\vdash\Delta}\mathsf{r}_1 \quad \dfrac{\vdash\Gamma}{\vdash\Delta}\mathsf{r}_1}{\vdash\Delta}\mathsf{sum}$$

$$\dfrac{\dfrac{\vdash\Gamma \quad \vdash\Gamma}{\vdash\Gamma}\mathsf{sum} \quad \vdash\Delta}{\vdash\Sigma}\mathsf{r}_2 \equiv \dfrac{\dfrac{\vdash\Gamma \quad \vdash\Delta}{\vdash\Sigma}\mathsf{r}_2 \quad \dfrac{\vdash\Gamma \quad \vdash\Delta}{\vdash\Sigma}\mathsf{r}_2}{\vdash\Sigma}\mathsf{sum} \equiv \dfrac{\vdash\Gamma \quad \dfrac{\vdash\Delta \quad \vdash\Delta}{\vdash\Delta}\mathsf{sum}}{\vdash\Sigma}\mathsf{r}_2 \qquad \dfrac{\vdash\Gamma \quad \dfrac{\vdash\Gamma \quad \vdash\Gamma}{\vdash\Gamma}\mathsf{sum}}{\vdash\Gamma}\mathsf{sum} \equiv \dfrac{\dfrac{\vdash\Gamma \quad \vdash\Gamma}{\vdash\Gamma}\mathsf{sum} \quad \vdash\Gamma}{\vdash\Gamma}\mathsf{sum}$$

Figure 2: The equivalence $\equiv$ on derivations generated by the rules zero and sum in DiLL_0, where r_1 is any unary rule in DiLL_0, and r_2 is any binary rule in DiLL_0 but sum.

Cut elimination. Rewriting rules $\rightsquigarrow_{\mathsf{cut}}$ for cut elimination in DiLL_0 sequent calculus are defined in Appendix B. They are just the formulation in the sequent calculus formalism of the cut-elimination steps defined and studied in [6, 17, 9] within the interaction nets formalism. With these "resource-sensitive" cut elimination steps it has been proved that the rule cut is admissible in DiLL_0.

Theorem 2.4 (Cut-elimination, [6, 17, 9])**.** *For every derivation π in DiLL_0 with conclusion $\vdash\Gamma$, there exists a cut-free derivation π' in DiLL_0 with conclusion $\vdash\Gamma$ such that $\pi \rightsquigarrow^*_{\mathsf{cut}} \pi'$.*

Note that if $\pi \rightsquigarrow_{\mathsf{cut}} \pi'$ with π canonical then π' is not necessarily canonical (*e.g.* if in π a cut ?c/!d or ?d/!w is above another rule), but π' can be rewritten in a canonical form (see Fact 2.2 above).

3 Calculus of structures for DiLL_0

In this section we introduce a deep inference system [14, 12] suitable for DiLL_0, using the *open deduction* formalism [13]. As a first novelty, we internalize the rules zero and sum of DiLL_0 sequent calculus at the level of formulas. Thus, *formulas* are defined by the following grammar:

$$A, B ::= a \mid \bar{a} \mid A \otimes B \mid A \parr B \mid 1 \mid \bot \mid !A \mid ?B \mid 0 \mid A + B$$

where $a, b, c, \ldots$ range over a countably infinite set of propositional variables (so, a MELL formula is a formula with no occurrences of $+$ and 0). Consider the least congruence $\simeq$ on formulas generated by:

$$\begin{array}{ccc}
A \parr (B \parr C) \simeq (A \parr B) \parr C & A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C & A + (B + C) \simeq (A + B) + C \\
A \parr B \simeq B \parr A & A \otimes B \simeq B \otimes A & A + B \simeq B + A \\
A \parr \bot \simeq A & A \otimes 1 \simeq A & A + 0 \simeq A \\
\multicolumn{3}{c}{A \parr (B + C) \simeq (A \parr B) + (A \parr C) \qquad A \otimes (B + C) \simeq (A \otimes B) + (A \otimes C)} \\
\multicolumn{3}{c}{!(A + B) \simeq\, !A + !B \qquad ?(A + B) \simeq\, ?A + ?B} \\
\multicolumn{3}{c}{0 \otimes A \simeq 0 \qquad 0 \parr A \simeq 0 \qquad ?0 \simeq 0 \qquad !0 \simeq 0}
\end{array} \tag{1}$$

With respect to $\simeq$, the formula 0 plays the role of annihilating element with respect all other connectives but $+$, for which it is a neutral element; every connective other than $+$ distributes over $+$.

A formula A is in *additive normal form* if it is a sum of MELL formulas, *i.e.* $A = A_1 + \cdots + A_n$ for some $n \in \mathbb{N}$, where all A_i's are MELL formulas ($A = 0$ for $n = 0$). For any $n \in \mathbb{N}$, we set $\mathsf{n} = \underbrace{1 + \cdots + 1}_{n \text{ times}}$.

A *context* (resp. MELL *context*) $\Omega\{\}$ is a formula (resp. MELL formula) with exactly one occurrence of the hole $\{\}$ (a special propositional variable). We write $\Omega\{A\}$ for the formula obtained from the context $\Omega\{\}$ by replacing its hole with the formula A.

Remark 3.1 (Additive normal form)**.** By definition of $\simeq$, if $\Omega\{\}$ is a context, then $\Omega\{A + 0\} \simeq \Omega\{A\}$ and $\Omega\{A + B\} \simeq \Omega\{A\} + \Omega\{B\}$. If $\Omega\{\}$ is a MELL context, $\Omega\{0\} \simeq 0$. In general, for every formula A there is an additive normal form A' such that $A \simeq A'$. Indeed, equivalences in (1) but the ones on the second line can be oriented to define a terminating rewriting relation whose normal forms are additive normal.

$$
\begin{array}{cccccccccccc}
\mathsf{ai}^{\downarrow}\frac{1}{a \parr \bar{a}} & \mathsf{!d}^{\downarrow}\frac{A}{!A} & \mathsf{?d}^{\downarrow}\frac{A}{?A} & \mathsf{!w}^{\downarrow}\frac{1}{!A} & \mathsf{?w}^{\downarrow}\frac{\bot}{?A} & \mathsf{!c}^{\downarrow}\frac{!A \otimes !A}{!A} & \mathsf{?c}^{\downarrow}\frac{?A \parr ?A}{?A} & +^{\downarrow}\frac{A+A}{A} & 0^{\downarrow}\frac{0}{A} & \multirow{2}{*}{$\mathsf{s}\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$} & \multirow{2}{*}{$\simeq\frac{A}{B}$} \\
\mathsf{ai}^{\uparrow}\frac{a \otimes \bar{a}}{\bot} & \mathsf{!d}^{\uparrow}\frac{?A}{A} & \mathsf{?d}^{\uparrow}\frac{!A}{A} & \mathsf{!w}^{\uparrow}\frac{?A}{\bot} & \mathsf{?w}^{\uparrow}\frac{!A}{1} & \mathsf{!c}^{\uparrow}\frac{?A}{?A \otimes ?A} & \mathsf{?c}^{\uparrow}\frac{!A}{!A \otimes !A} & +^{\uparrow}\frac{A}{A+A} & 0^{\uparrow}\frac{A}{0} & &
\end{array}
$$

Figure 3: The rules of the deep inference system SDDI (A, B, C are MELL formulas, except in the rule $\simeq$ where A and B are formulas—possibly not MELL formulas—such that $A \simeq B$).

Derivations. A *deep inference system* $\mathcal{S}$ is given by a set of inference rules. A *derivation* $\mathcal{D}$ from a *premise* B to a *conclusion* A in a deep inference system $\mathcal{S}$ is written $\begin{array}{c} B \\ \mathcal{D}\|\mathcal{S} \\ A \end{array}$ and defined inductively by:

- a formula A is a derivation (denoted by A) with premise and conclusion A;
- if for all $i \in \{1,2\}$ $\mathcal{D}_i$ is a derivation with premise B_i and conclusion A_i, then for any $\bullet \in \{\parr, \otimes, +\}$ $\mathcal{D}_1 \bullet \mathcal{D}_2$ is a derivation with premise $B_1 \bullet B_2$ and conclusion $A_1 \bullet A_2$ (see (2) below on the left);
- if $\rho\frac{A_1}{B_2} \in \mathcal{S}$ and, for all $i \in \{1,2\}$, $\mathcal{D}_i$ is a derivation with premise B_i and conclusion A_i, then $\mathcal{D}_1 \circ_\rho \mathcal{D}_2$ is a derivation with premise B_1 and conclusion A_2 (see (2) below on the right).

$$
\begin{array}{c} B_1 \bullet B_2 \\ \mathcal{D}_1 \bullet \mathcal{D}_2 \| \mathcal{S} \\ A_1 \bullet A_2 \end{array} = \boxed{\begin{array}{c} B_1 \\ \mathcal{D}_1 \| \mathcal{S} \\ A_1 \end{array}} \bullet \boxed{\begin{array}{c} B_2 \\ \mathcal{D}_2 \| \mathcal{S} \\ A_2 \end{array}} \quad \text{for } \bullet \in \{\parr, \otimes, +\} \qquad \begin{array}{c} B_1 \\ \mathcal{D}_1 \circ_\rho \mathcal{D}_2 \| \mathcal{S} \\ A_2 \end{array} = \rho\,\frac{\begin{array}{c} B_1 \\ \mathcal{D}_1 \| \mathcal{S} \\ A_1 \end{array}}{\begin{array}{c} B_2 \\ \mathcal{D}_2 \| \mathcal{S} \\ A_2 \end{array}} \quad \text{for } \rho \in \mathcal{S} \tag{2}
$$

We write $B \overset{\mathcal{S}}{\vdash} A$ if there is a derivation in $\mathcal{S}$ from B to A. A rule $\rho\frac{B}{A}$ is *derivable* in $\mathcal{S}$ if $B \overset{\mathcal{S}}{\vdash} A$.

The system SDDI is defined by the rules in Figure 3. The *down-fragment* and *up-fragment*[2] of SDDI are the sets of rules $\mathsf{DDI}^{\downarrow} = \{\mathsf{ai}^{\downarrow}, \mathsf{!d}^{\downarrow}, \mathsf{?d}^{\downarrow}, \mathsf{!w}^{\downarrow}, \mathsf{?w}^{\downarrow}, \mathsf{!c}^{\downarrow}, \mathsf{?c}^{\downarrow}, +^{\downarrow}, 0^{\downarrow}, \mathsf{s}, \simeq\}$ and $\mathsf{DDI}^{\uparrow} = \{\mathsf{ai}^{\uparrow}, \mathsf{!d}^{\uparrow}, \mathsf{?d}^{\uparrow}, \mathsf{!w}^{\uparrow}, \mathsf{?w}^{\uparrow}, \mathsf{!c}^{\uparrow}, \mathsf{?c}^{\uparrow}, +^{\uparrow}, 0^{\uparrow}, \mathsf{s}, \simeq\}$, respectively; we set $\mathsf{DDI}^{\downarrow}_{-} = \mathsf{DDI}^{\downarrow} \setminus \{+^{\downarrow}, 0^{\downarrow}\}$. All rules in SDDI have exactly one premise. Note the perfect symmetry between $\mathsf{DDI}^{\downarrow}$ and $\mathsf{DDI}^{\uparrow}$, and that $\mathsf{SDDI} = \mathsf{DDI}^{\downarrow} \cup \mathsf{DDI}^{\uparrow}$. Observe that in a derivation in $\mathsf{DDI}^{\downarrow}_{-}$ only MELL formulas occur.

Remark 3.2 (Deep)**.** The idea of deep inference is that inference rules can be applied in any context. Said differently, in a deep inference system $\mathcal{S}$, if $\rho\frac{B}{A} \in \mathcal{S}$ then for any context $\Omega\{\}$ $\rho\frac{\Omega\{B\}}{\Omega\{A\}}$ is derivable in $\mathcal{S}$. In this way, a derivation in $\mathcal{S}$ can be seen as a finite *sequence* of "deep" inference rules. For instance, the derivation $\boxed{\mathsf{!d}^{\downarrow}\frac{a}{!a}} \otimes \boxed{\mathsf{!d}^{\downarrow}\frac{b}{!b}}$ in $\mathsf{DDI}^{\downarrow}$ can be "sequenced" as both $\mathsf{!d}^{\downarrow}\frac{\mathsf{!d}^{\downarrow}\frac{a \otimes b}{a \otimes !b}}{!a \otimes !b}$ and $\mathsf{!d}^{\downarrow}\frac{\mathsf{!d}^{\downarrow}\frac{a \otimes b}{!a \otimes b}}{!a \otimes !b}$.

System SDDI has only the *atomic* introduction rules $\mathsf{ai}^{\downarrow}$ and $\mathsf{ai}^{\uparrow}$ (a is a propositional variable in Figure 3): they can be interpreted as the atomic version of ax- and cut-rules of sequent calculus, respectively. The general (non-atomic) versions of the rules $\mathsf{ai}^{\downarrow}$ and $\mathsf{ai}^{\uparrow}$ are respectively:

$$
\mathsf{i}^{\downarrow}\frac{1}{A \parr \bar{A}} \qquad \mathsf{i}^{\uparrow}\frac{A \otimes \bar{A}}{\bot} \qquad \text{(where } A \text{ is a MELL formula)}
$$

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[2] Usually in the literature on deep inference, the dual rule $\mathsf{r}^{\uparrow}$ of a rule $\mathsf{r}^{\downarrow}$ is called "co-r". We avoid these names because they clash with the usual terminology in the literature on DiLL_0, see Footnote 1.

However, the rules $\mathsf{i}^{\downarrow}$ and $\mathsf{i}^{\uparrow}$ are derivable in SDDI (Lemma 3.3). Derivability of $\mathsf{i}^{\downarrow}$ corresponds to Proposition 2.3 in $\mathsf{DiLL_0}$ sequent calculus, but derivability of $\mathsf{i}^{\uparrow}$ is a typical result in deep inference systems that does not have a corresponding result in $\mathsf{DiLL_0}$ sequent calculus.

Claim p. 6
Proof p. 11

Lemma 3.3 (Atomic axioms and atomic cuts)**.** *The rule* $\mathsf{i}^{\downarrow}$ *is derivable in* $\{\mathsf{ai}^{\downarrow},\mathsf{s},?\mathsf{d}^{\downarrow},!\mathsf{d}^{\downarrow},\simeq\}$*; and the rule* $\mathsf{i}^{\uparrow}$ *is derivable in* $\{\mathsf{ai}^{\uparrow},\mathsf{s},?\mathsf{d}^{\uparrow},!\mathsf{d}^{\uparrow},\simeq\}$.

The rule $\mathsf{i}^{\uparrow}$ plays a special role in deep inference systems, as the cut does in sequent calculi. In particular, it makes the rules in $\mathsf{DDI}^{\uparrow}$ (second line in Figure 3) superfluous. Note that $\mathsf{ai}^{\uparrow}$ is not enough for that, because $\mathsf{i}^{\uparrow}$ needs $!\mathsf{d}^{\uparrow}$ and $?\mathsf{d}^{\uparrow}$ to be simulated by $\mathsf{ai}^{\uparrow}$ (Lemma 3.3).

Claim p. 6
Proof p. 11

Proposition 3.4 (Getting rid of up-rules via $\mathsf{i}^{\uparrow}$)**.**

1. *Any* $\rho^{\uparrow}\in\{!\mathsf{d}^{\uparrow},?\mathsf{d}^{\uparrow},!\mathsf{c}^{\uparrow},?\mathsf{c}^{\uparrow},!\mathsf{w}^{\uparrow},?\mathsf{w}^{\uparrow}\}$ *is derivable in* $\{\rho^{\downarrow},\mathsf{i}^{\uparrow},\mathsf{i}^{\downarrow},\mathsf{s},\simeq\}$, *and* $+^{\uparrow}$ *is derivable in* $\{\mathsf{0}^{\downarrow},\simeq\}$.
2. Down fragment plus $\mathsf{i}^{\uparrow},\mathsf{0}^{\uparrow}$*: For any formula A and* $n\in\mathbb{N}$, *one has* $\mathsf{n} \overset{\mathsf{DDI}^{\downarrow}\cup\{\mathsf{i}^{\uparrow},\mathsf{0}^{\uparrow}\}}{\longmapsto} A$ *if and only if* $\mathsf{n} \overset{\mathsf{SDDI}}{\longmapsto} A$.
3. SDDI-Canonicity*: For any* MELL *formula A and* $n\in\mathbb{N}$, *if* $\mathsf{n} \overset{\mathsf{SDDI}}{\longmapsto} A$, *then either* $1 \overset{\mathsf{DDI}^{\downarrow}_{-}\cup\{\mathsf{i}^{\uparrow}\}}{\longmapsto} A$ *or* $0 \overset{\{\mathsf{0}^{\downarrow}\}}{\longmapsto} A$.

4 Correspondence between $\mathsf{DiLL_0}$ and SDDI

In this section we prove that SDDI is a sound and complete proof system for $\mathsf{DiLL_0}$ sequent calculus. At first sight, this result is obvious because the rules zero in $\mathsf{DiLL_0}$ and $\mathsf{0}^{\downarrow}$ in SDDI make everything provable. But the interest is to show that the fragments without zero and $\mathsf{0}^{\downarrow}$ correspond each other.

If $\Gamma = A_1,\ldots,A_n$ is a MELL sequent, we set $[\![\Gamma]\!] = A_1 \parr \cdots \parr A_n$ (in particular, $[\![\Gamma]\!] = \bot$ for $n = 0$).

Theorem 4.1 (Completeness)**.** *Let* Γ *be a* MELL *sequent. If* $\overset{\mathsf{DiLL_0}}{\longmapsto} \Gamma$ *then* $\mathsf{n} \overset{\mathsf{DDI}^{\downarrow}\cup\{\mathsf{i}^{\uparrow}\}}{\longmapsto} [\![\Gamma]\!]$ *and* $\mathsf{n} \overset{\mathsf{SDDI}}{\longmapsto} [\![\Gamma]\!]$ *for some* $n\in\mathbb{N}$. *Moreover,*

1. slice vs. zero*: if* $\overset{\mathsf{DiLL_0^-}}{\longmapsto} \Gamma$ *then* $1 \overset{\mathsf{DDI}^{\downarrow}_{-}\cup\{\mathsf{i}^{\uparrow}\}}{\longmapsto} [\![\Gamma]\!]$ *and* $1 \overset{\mathsf{SDDI}}{\longmapsto} [\![\Gamma]\!]$*; if* $\overset{\{\mathsf{zero}\}}{\longmapsto} \Gamma$ *then* $0 \overset{\{\mathsf{0}^{\downarrow}\}}{\longmapsto} [\![\Gamma]\!]$*;*
2. cut-free*: if* $\overset{\mathsf{DiLL_0}\setminus\{\mathsf{cut}\}}{\longmapsto} \Gamma$ *then* $\mathsf{n} \overset{\mathsf{DDI}^{\downarrow}}{\longmapsto} [\![\Gamma]\!]$ *for some* $n\in\mathbb{N}$.

Proof. By Proposition 3.4.2, it suffices to show that if $\overset{\mathsf{DiLL_0}}{\longmapsto} \Gamma$ then $\mathsf{n} \overset{\mathsf{DDI}^{\downarrow}\cup\{\mathsf{i}^{\uparrow}\}}{\longmapsto} [\![\Gamma]\!]$ for some $n\in\mathbb{N}$. If $\overset{\mathsf{DiLL_0}}{\longmapsto} \Gamma$ then there is a derivation π of Γ in $\mathsf{DiLL_0}$ with atomic axioms (Proposition 2.3). By induction on π, we define a derivation $[\![\pi]\!]$ in $\mathsf{DDI}^{\downarrow}\cup\{\mathsf{i}^{\uparrow}\}$ from n to $[\![\Gamma]\!]$, as shown in Figure 4, for some $n\in\mathbb{N}$. If, moreover, π is a slice (resp. cut-free), then $[\![\pi]\!]$ is a derivation in $\mathsf{DDI}^{\downarrow}_{-}\cup\{\mathsf{i}^{\uparrow}\}$ from 1 (resp. a derivation in $\mathsf{DDI}^{\downarrow}$ from n), according to the translation in Figure 4. Clearly, the rule zero is translated by $\mathsf{0}^{\downarrow}$. ☐

Not only each slice of a $\mathsf{DiLL_0}$ derivation corresponds to a SDDI derivation with only MELL formulas (completeness), but also the converse holds (soundness).

Theorem 4.2 (Soundness)**.** *For any* MELL *sequent* Γ *and* $n\in\mathbb{N}$, *if* $\mathsf{n} \overset{\mathsf{DDI}^{\downarrow}\cup\{\mathsf{i}^{\uparrow},\mathsf{0}^{\uparrow}\}}{\longmapsto} [\![\Gamma]\!]$ *(or equivalently* $\mathsf{n} \overset{\mathsf{SDDI}}{\longmapsto} [\![\Gamma]\!]$*) then* $\overset{\mathsf{DiLL_0}}{\longmapsto} \Gamma$ *and more precisely, either* $1 \overset{\mathsf{DDI}^{\downarrow}_{-}\cup\{\mathsf{i}^{\uparrow}\}}{\longmapsto} [\![\Gamma]\!]$ *and* $\overset{\mathsf{DiLL_0^-}}{\longmapsto} \Gamma$, *or* $0 \overset{\{\mathsf{0}^{\downarrow}\}}{\longmapsto} [\![\Gamma]\!]$ *and* $\overset{\{\mathsf{zero}\}}{\longmapsto} \Gamma$.

Proof. By Proposition 3.4.3, as $\mathsf{n} \overset{\mathsf{DDI}^{\downarrow}\cup\{\mathsf{i}^{\uparrow},\mathsf{0}^{\uparrow}\}}{\longmapsto} [\![\Gamma]\!]$ and $[\![\Gamma]\!]$ is a MELL formula, either $0 \overset{\{\mathsf{0}^{\downarrow}\}}{\longmapsto} [\![\Gamma]\!]$ or $1 \overset{\mathsf{DDI}^{\downarrow}_{-}\cup\{\mathsf{i}^{\uparrow}\}}{\longmapsto} [\![\Gamma]\!]$.

In the first case, one can build the following derivation in $\mathsf{DiLL_0}$ sequent calculus: $\dfrac{}{\vdash \Gamma}\,\mathsf{zero}$.

In the second case, there is a derivation $\mathcal{D}$ in $\mathsf{DDI}^{\downarrow}_{-}\cup\{\mathsf{i}^{\uparrow}\}$ from 1 to $[\![\Gamma]\!]$ with only $\simeq$-rules involving MELL formulas. For any $\rho\in\mathsf{DDI}^{\downarrow}_{-}\cup\{\mathsf{i}^{\uparrow}\}$, if $\rho\,\dfrac{B}{A}$ then $\overset{\mathsf{DiLL_0}}{\longmapsto} \overline{B},A$ as shown in Figure 5 (we omit when $\rho = \simeq$, as it is standard). Then, by induction on the MELL context $\Omega\{\}$, if $\rho\,\dfrac{\Omega\{B\}}{\Omega\{A\}}$ then $\overset{\mathsf{DiLL_0}}{\longmapsto} \overline{\Omega\{B\}},\Omega\{A\}$.

We define a derivation of $\vdash [\![\Gamma]\!]$ in $\mathsf{DiLL_0^-}$ by induction on the number of rules in $\mathcal{D}$ as follows:

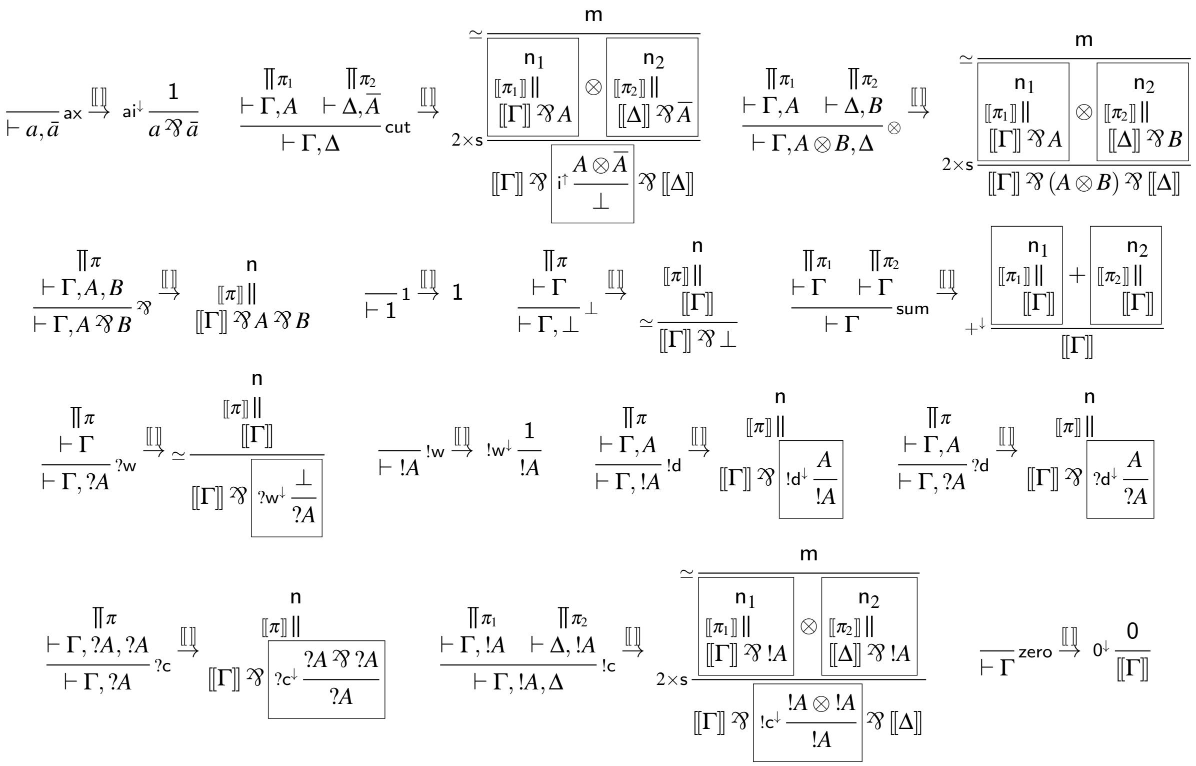

Figure 4: Translation of DiLL_0 sequent calculus derivations into $\mathsf{DDI}^{\downarrow} \cup \{\mathsf{i}^{\uparrow}\}$ derivations ($m = n_1 \times n_2$).

$$1 \;\to\; \frac{}{\vdash 1}\,1 \qquad\qquad \rho\,\frac{\overset{\mathsf{n}}{\underset{[\![\Delta]\!]}{\Vert \mathsf{DDI}^{\downarrow}\cup\{\mathsf{i}^{\uparrow}\}}}}{[\![\Gamma]\!]} \;\to\; \frac{\overset{HI\,\Pi\,\mathsf{DiLL}_0}{\vdash [\![\Delta]\!]} \qquad \overset{\Pi\,\mathsf{DiLL}_0}{\vdash \overline{[\![\Delta]\!]}, [\![\Gamma]\!]}}{\vdash [\![\Gamma]\!]}\,\mathsf{cut}$$

By reversibility of $⅋$ (if $\overset{\mathsf{DiLL}_0^-}{\vdash} A ⅋ B$ then $\overset{\mathsf{DiLL}_0^-}{\vdash} A, B$), also $\vdash \Gamma$ is provable in DiLL_0 sequent calculus. ☐

Theorem 4.3 (Sequent calculus vs. deep inference)**.** *Let* Γ *be a* MELL *sequent.*

1. DiLL_0 vs. SDDI*:* $\overset{\mathsf{DiLL}_0}{\vdash} \Gamma$ *if and only if* $\mathsf{n} \overset{\mathsf{SDDI}}{\vdash} [\![\Gamma]\!]$ *for some* $n \in \mathbb{N}$.
2. DiLL_0 cut-free vs. $\mathsf{DDI}^{\downarrow}$*:* $\overset{\mathsf{DiLL}_0\setminus\{\mathsf{cut}\}}{\vdash} \Gamma$ *if and only if* $\mathsf{n} \overset{\mathsf{DDI}^{\downarrow}}{\vdash} [\![\Gamma]\!]$ *for some* $n \in \mathbb{N}$.

Proof. 1. Consequence of completeness (Theorem 4.1, for ⇒) and soundness (Theorem 4.2, for ⇐).

2. For ⇒, see Theorem 4.1.2. The converse (⇐) follows from Theorems 2.4 and 4.3.1. ☐

Corollary 4.4 (Up-fragment elimination)**.** *For any formula A and* $n \in \mathbb{N}$*, if* $\mathsf{n} \overset{\mathsf{SDDI}}{\vdash} A$ *then* $\mathsf{n} \overset{\mathsf{DDI}^{\downarrow}}{\vdash} A$.

Corollary 4.4 is the deep inference version of cut-elimination, since in $\mathsf{DDI}^{\downarrow}$ there is no analogue of the cut rule. It follows from Theorem 4.3.1-2 and DiLL_0 sequent calculus cut-elimination (Theorem 2.4). In the next section we provide the tool to give an internal proof of this result in our system (Theorem 4.6).

$$\mathsf{ai}^{\downarrow}\dfrac{1}{a \parr \bar{a}} \to \dfrac{\dfrac{\dfrac{}{\vdash a,\bar{a}}\,\mathsf{ax}}{\vdash a \parr \bar{a}}\,\parr}{\vdash a \parr \bar{a},\bot}\,\bot \qquad \mathsf{i}^{\uparrow}\dfrac{A \otimes \overline{A}}{\bot} \to \dfrac{\dfrac{\dfrac{}{\vdash \overline{A},A}\,\mathsf{ax}}{\vdash \overline{A} \parr A}\,\parr}{\vdash \overline{A} \parr A,\bot}\,\bot \qquad \mathsf{!w}^{\downarrow}\dfrac{1}{!A} \to \dfrac{\dfrac{}{\vdash !A}\,\mathsf{!w}}{\vdash !A,\bot}\,\bot \qquad \mathsf{?w}^{\downarrow}\dfrac{\bot}{?A} \to \dfrac{\dfrac{}{\vdash 1}\,1}{\vdash 1, ?A}\,\mathsf{?w}$$

$$\mathsf{?d}^{\downarrow}\dfrac{A}{?A} \to \dfrac{\dfrac{}{\vdash \overline{A},A}\,\mathsf{ax}}{\vdash \bar{A}, ?A}\,\mathsf{?d} \qquad \mathsf{!d}^{\downarrow}\dfrac{A}{!A} \to \dfrac{\dfrac{}{\vdash \overline{A},A}\,\mathsf{ax}}{\vdash \overline{A}, !A}\,\mathsf{!d} \qquad \mathsf{s}\dfrac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \to \dfrac{\dfrac{\dfrac{}{\vdash \overline{A},A}\,\mathsf{ax} \quad \dfrac{\dfrac{}{\vdash \overline{B},B}\,\mathsf{ax} \quad \dfrac{}{\vdash \overline{C},C}\,\mathsf{ax}}{\vdash B,C,\overline{B} \otimes \overline{C}}\,\otimes}{\vdash A \otimes B, C, \overline{A}, \overline{B} \otimes \bar{C}}\,\otimes}{\vdash (A \otimes B) \parr C, \overline{A} \parr (\overline{B} \otimes \overline{C})}\,2\times\parr$$

$$\mathsf{?c}^{\downarrow}\dfrac{?A \parr ?A}{?A} \to \dfrac{\dfrac{\dfrac{}{\vdash !\overline{A}, ?A}\,\mathsf{ax} \quad \dfrac{}{\vdash !\overline{A}, ?A}\,\mathsf{ax}}{\vdash !\overline{A} \otimes !\overline{A}, ?A, ?A}\,\otimes}{\vdash !\overline{A} \otimes !\overline{A}, ?A}\,\mathsf{?c} \qquad \mathsf{!c}^{\downarrow}\dfrac{!A \otimes !A}{!A} \to \dfrac{\dfrac{\dfrac{}{\vdash ?\overline{A}, !A}\,\mathsf{ax} \quad \dfrac{}{\vdash ?\overline{A}, !A}\,\mathsf{ax}}{\vdash ?\overline{A}, ?\overline{A}, !A}\,\mathsf{!c}}{\vdash !A, ?\overline{A} \parr ?\overline{A}}\,\parr$$

Figure 5: Interpretation of the rules in $\mathsf{DDI}^{\downarrow}_{-} \cup \{\mathsf{i}^{\uparrow}\}$ as derivations in DiLL_0 sequent calculus.

Normalization in SDDI. We define a standard form for derivations in SDDI and a normalization procedure. This result is given by defining some *rule permutation* as done for MELL in [20, 14]. In fact, in SDDI we cannot use Guglielmi and Tubella's normalization result [21, 22] for open deduction *splittable* systems: this is due to the presence in our syntax of the connective $+$ and its unit 0. So, we define rule permutation by relying on the rules for the connective $+$ and its unit 0. This behavior is coherent with the dynamics of cut elimination in DiLL_0 [6, 17, 9].

Due to the more flexible structure of our syntax, we can explicitly observe the process of *slices duplication*, implicit in DiLL_0 sequent calculus. Moreover, thanks to the rules symmetry in SDDI, we are able to observe three kinds of rules commutations corresponding to MLL *cut-elimination steps* (involving $\mathsf{ai}^{\uparrow}$, $\mathsf{ai}^{\downarrow}$ and s only), *resource management steps* (involving the ?- and the !-rules only) and *slices operations*.

Lemma 4.5 (SDDI-Decomposition)**.** *If* $B \overset{\mathsf{SDDI}}{\vdash} A$*, then there is a derivation (called* standard form*) in* SDDI *from* B *to* A *of the following form (for some formulas* $B', B'', B''', B'''', A'''', A''', A'', A'$*):*

$$B \overset{\{0^{\uparrow},+^{\uparrow}\}}{\vdash} B' \overset{\{\mathsf{?w}^{\uparrow},\mathsf{?c}^{\uparrow},\mathsf{!c}^{\uparrow}\}}{\vdash} B'' \overset{\{\mathsf{?d}^{\uparrow},\mathsf{!d}^{\uparrow}\}}{\vdash} B''' \overset{\{\mathsf{ai}^{\downarrow},\mathsf{!w}^{\downarrow}\}}{\vdash} B'''' \overset{\{\mathsf{s}\}}{\vdash} A'''' \overset{\{\mathsf{ai}^{\uparrow},\mathsf{!w}^{\uparrow}\}}{\vdash} A''' \overset{\{\mathsf{?d}^{\downarrow},\mathsf{!d}^{\downarrow}\}}{\vdash} A''' \overset{\{\mathsf{?w}^{\downarrow},\mathsf{?c}^{\downarrow},\mathsf{!c}^{\downarrow}\}}{\vdash} A' \overset{\{0^{\downarrow},+^{\downarrow}\}}{\vdash} A.$$

Proof (sketch): Using rule permutations and formula equivalence, we push the $+^{\uparrow}$ and $0^{\uparrow}$ up and $+^{\downarrow}$ and $0^{\downarrow}$ down in the derivation. This yields a derivation as a sum of sum-free derivations. For each of these derivations, we define normalization steps in a similar way that was done in [20, 14]. ☐

Lemma 4.5 and the fact that 0 is the unit for $+$ allow us to give an *internal* proof of Corollary 4.4.

Theorem 4.6 (Normalization)**.** *If* A *is a formula and* $n \in \mathbb{N}$*, then* $\mathsf{n} \overset{\mathsf{SDDI}}{\vdash} A$ *if and only if* $\mathsf{n} \overset{\mathsf{DDI}^{\downarrow}}{\vdash} A$.

The internal normalization procedure for SDDI to prove Theorem 4.6 provides derivations in $\mathsf{DDI}^{\downarrow}$, the "cut-free" fragment of SDDI, and the translation defined in Figure 4 sends cut-free DiLL_0 derivations into $\mathsf{DDI}^{\downarrow}$ (Theorem 4.3.2). A natural question arises: does normalization in SDDI correspond to cut elimination in DiLL_0 sequent calculus? In other words, does the translation in Figure 4 commute with (sequent calculus and deep inference) normalization? The answer is negative: there is a derivation π in DiLL_0 sequent calculus that reduces to a cut-free derivation π_0 via cut elimination, but its translation $[\![\pi]\!]$ in SDDI normalizes to a $\mathsf{DDI}^{\downarrow}$ derivation other that $[\![\pi_0]\!]$. As an ongoing work, we conjecture that a refinement of the translation in Figure 4 does *commute* with normalization.

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Technical appendix

A Omitted proofs in Sections 2 to 4

Proposition 2.3 (Atomic axioms)**.** *For every derivation π in DiLL_0 with conclusion $\vdash \Gamma$, there exists a derivation π' in DiLL_0 with conclusion $\vdash \Gamma$ and atomic axioms. If, moreover, π is canonical (resp. a slice) then π' is canonical (resp. a slice).*

Claim p. 3
Proof p. 11

Proof. Rewrite any non-atomic instance of the rule ax according to the relation $\rightsquigarrow_\eta$ below:

$$\dfrac{}{\vdash A\otimes B,\overline{A}\parr\overline{B}}\,\mathsf{ax} \rightsquigarrow_\eta \dfrac{\dfrac{\dfrac{}{\vdash A,\overline{A}}\,\mathsf{ax}\qquad\dfrac{}{\vdash B,\overline{B}}\,\mathsf{ax}}{\vdash A\otimes B,\overline{A},\overline{B}}\,\otimes}{\vdash A\otimes B,\overline{A}\parr\overline{B}}\,\parr \qquad \dfrac{}{\vdash 1,\bot}\,\mathsf{ax} \rightsquigarrow_\eta \dfrac{\dfrac{}{\vdash 1}\,1}{\vdash 1,\bot}\,\bot \qquad \dfrac{}{\vdash\, !A,?\overline{A}}\,\mathsf{ax} \rightsquigarrow_\eta \dfrac{\dfrac{\dfrac{}{\vdash A,\overline{A}}\,\mathsf{ax}}{\vdash ?A,\overline{A}}\,?\mathsf{d}}{\vdash ?A,!\overline{A}}\,!\mathsf{d}$$

It is immediate to prove that the relation $\rightsquigarrow_\eta$ on the derivations of DiLL_0 is terminating.

Note that $\rightsquigarrow_\eta$ does not introduce any rule sum or zero, hence if $\pi \rightsquigarrow_\eta^* \pi'$ where π is canonical or a slice, then π' is canonical or a slice, respectively. $\square$

Lemma 3.3 (Atomic axioms and atomic cuts)**.** *The rule $\mathsf{i}^\downarrow$ is derivable in $\{\mathsf{ai}^\downarrow,\mathsf{s},?\mathsf{d}^\downarrow,!\mathsf{d}^\downarrow,\simeq\}$; and the rule $\mathsf{i}^\uparrow$ is derivable in $\{\mathsf{ai}^\uparrow,\mathsf{s},?\mathsf{d}^\uparrow,!\mathsf{d}^\uparrow,\simeq\}$.*

Claim p. 6
Proof p. 11

Proof. Concerning $\mathsf{i}^\downarrow$, the proof is by induction on the size of the MELL formula A in $\mathsf{i}^\downarrow\dfrac{1}{A\parr\bar{A}}$. Cases:

- if $A = a$ is a propositional variable (and similarly if $A = \bar{a}$), then $\mathsf{ai}^\downarrow\dfrac{1}{a\parr\bar{a}}$;
- if $A = 1$ (and similarly if $A = \bot$), then $\simeq\dfrac{1}{1\parr\bot}$;
- if $A = B\otimes C$ (and similarly for $A = B\parr C$), then $\simeq\dfrac{1}{2{\times}\mathsf{s}\dfrac{\boxed{\begin{matrix}1\\ \Vert\,\mathsf{IH}\\ B\parr\bar{B}\end{matrix}}\otimes\boxed{\begin{matrix}1\\ \Vert\,\mathsf{IH}\\ C\parr\bar{C}\end{matrix}}}{(B\otimes C)\parr(\bar{B}\parr\bar{C})}}$;
- if $A = !B$ (and similarly if $A = ?B$), then $\begin{matrix}1\\ \Vert\,\mathsf{IH}\\ \boxed{!\mathsf{d}^\downarrow\dfrac{B}{!B}}\parr\boxed{?\mathsf{d}^\downarrow\dfrac{\bar{B}}{?\bar{B}}}\end{matrix}$.

The proof for $\mathsf{i}^\uparrow$ is dual, using $\mathsf{ai}^\uparrow$, $!\mathsf{d}^\uparrow$, $?\mathsf{d}^\uparrow$ instead of $\mathsf{ai}^\downarrow$, $!\mathsf{d}^\downarrow$, $?\mathsf{d}^\downarrow$, respectively. $\square$

Proposition 3.4 (Getting rid of up-rules via $\mathsf{i}^\uparrow$)**.**

Claim p. 6
Proof p. 11

1. *Any $\rho^\uparrow \in \{!\mathsf{d}^\uparrow, ?\mathsf{d}^\uparrow, !\mathsf{c}^\uparrow, ?\mathsf{c}^\uparrow, !\mathsf{w}^\uparrow, ?\mathsf{w}^\uparrow\}$ is derivable in $\{\rho^\downarrow,\mathsf{i}^\uparrow,\mathsf{i}^\downarrow,\mathsf{s},\simeq\}$, and $+^\uparrow$ is derivable in $\{0^\downarrow,\simeq\}$.*
2. Down fragment plus $\mathsf{i}^\uparrow, 0^\uparrow$*: For any formula A and $n \in \mathbb{N}$, one has* $\mathsf{n} \overset{\mathsf{DDI}^\downarrow\cup\{\mathsf{i}^\uparrow,0^\uparrow\}}{\longmapsto} A$ *if and only if* $\mathsf{n} \overset{\mathsf{SDDI}}{\longmapsto} A$.
3. SDDI-Canonicity*: For any MELL formula A and $n \in \mathbb{N}$, if* $\mathsf{n} \overset{\mathsf{SDDI}}{\longmapsto} A$*, then either* $1 \overset{\mathsf{DDI}^{\underline{\downarrow}}\cup\{\mathsf{i}^\uparrow\}}{\longmapsto} A$ *or* $0 \overset{\{0^\downarrow\}}{\longmapsto} A$.

Proof. 1. For a rule $\rho^{\uparrow}\,\frac{\bar{B}}{\bar{A}} \in \{!\mathsf{d}^{\uparrow}, ?\mathsf{d}^{\uparrow}, !\mathsf{c}^{\uparrow}, ?\mathsf{c}^{\uparrow}, !\mathsf{w}^{\uparrow}, ?\mathsf{w}^{\uparrow}\}$, see (3) below. For $+^{\uparrow}\,\frac{A}{A+A}$ see (4) below.

$$\simeq\dfrac{\bar{B}}{\mathsf{s}\dfrac{\boxed{\mathsf{i}^{\downarrow}\dfrac{1}{\bar{A}\parr A}}\otimes\bar{B}}{\simeq\dfrac{\bar{A}\parr\boxed{\mathsf{i}^{\uparrow}\dfrac{\boxed{\rho^{\downarrow}\dfrac{A}{B}}\otimes\bar{B}}{\bot}}}{\bar{A}}}} \tag{3}$$

$$\simeq\dfrac{A}{\mathsf{0}^{\downarrow}\dfrac{\mathsf{0}}{A}+A} \tag{4}$$

2. If $\mathsf{n}\overset{\mathsf{DDI}^{\downarrow}\cup\{\mathsf{i}^{\uparrow},\mathsf{0}^{\uparrow}\}}{\vdash} A$ then $\mathsf{n}\overset{\mathsf{SDDI}}{\vdash} A$ because any rule in $\mathsf{DDI}^{\downarrow}\cup\{\mathsf{0}^{\uparrow}\}$ is in SDDI, and $\mathsf{i}^{\uparrow}$ is derivable in SDDI (Lemma 3.3). Conversely, in a derivation in SDDI from n to A, any instance of a rule $\rho^{\uparrow} \in \{!\mathsf{d}^{\uparrow}, ?\mathsf{d}^{\uparrow}, !\mathsf{c}^{\uparrow}, ?\mathsf{c}^{\uparrow}, !\mathsf{w}^{\uparrow}, ?\mathsf{w}^{\uparrow}\}$ can be replaced by the rules $\mathsf{i}^{\downarrow}, \mathsf{i}^{\uparrow}, \rho^{\downarrow}, \mathsf{s}, \simeq$ (Proposition 3.4.1): $\mathsf{i}^{\downarrow}$ is derivable in $\mathsf{DDI}^{\downarrow}$ (Lemma 3.3), $\mathsf{i}^{\uparrow}, \rho^{\downarrow}, \mathsf{s}, \simeq$ are in $\mathsf{DDI}^{\downarrow}\cup\{\mathsf{i}^{\uparrow}\}$; any instance of the rule $+^{\uparrow}$ can be replaced by the rules $\mathsf{0}^{\downarrow}, \simeq$ (Proposition 3.4.1), which are in $\mathsf{DDI}^{\downarrow}$. Therefore, any derivation in SDDI from n to A can be rewritten as a derivation in $\mathsf{DDI}^{\downarrow}\cup\{\mathsf{i}^{\uparrow},\mathsf{0}^{\uparrow}\}$ from n to A.

3. Since $\mathsf{m}\overset{\mathsf{SDDI}}{\vdash} A$, there is a derivation $\mathcal{D}$ of $\mathsf{m}\overset{\mathsf{DDI}^{\downarrow}\cup\{\mathsf{i}^{\uparrow},\mathsf{0}^{\uparrow}\}}{\vdash} A$, by Proposition 3.4.2. Then in $\mathcal{D}$ we can push $+^{\downarrow}$ and $\mathsf{0}^{\downarrow}$ down and $\mathsf{0}^{\uparrow}$ up by means of the following permutations, where $\rho \in \mathsf{DDI}^{\underline{\downarrow}}\cup\{\mathsf{i}^{\uparrow}\}$:

$$+^{\downarrow}\dfrac{B+B}{\boxed{\rho\dfrac{B}{A}}} \rightsquigarrow +^{\downarrow}\dfrac{\boxed{\rho\dfrac{B}{A}}+\boxed{\rho\dfrac{B}{A}}}{A} \qquad \mathsf{0}^{\downarrow}\dfrac{\mathsf{0}}{\rho\dfrac{B}{A}} \rightsquigarrow \mathsf{0}^{\downarrow}\dfrac{\mathsf{0}}{A} \qquad +^{\downarrow}\dfrac{B+B}{\boxed{\mathsf{0}^{\uparrow}\dfrac{B}{\mathsf{0}}}} \rightsquigarrow +^{\downarrow}\dfrac{\boxed{\mathsf{0}^{\uparrow}\dfrac{B}{\mathsf{0}}}+\boxed{\mathsf{0}^{\uparrow}\dfrac{B}{\mathsf{0}}}}{\mathsf{0}} \qquad \mathsf{0}^{\downarrow}\dfrac{\mathsf{0}}{\mathsf{0}^{\uparrow}\dfrac{B}{\mathsf{0}}} \rightsquigarrow \mathsf{0} \qquad \rho\dfrac{B}{\mathsf{0}^{\uparrow}\dfrac{A}{\mathsf{0}}} \rightsquigarrow \mathsf{0}^{\uparrow}\dfrac{B}{\mathsf{0}}$$

We obtain a derivation $\mathsf{m}\overset{\{\mathsf{0}^{\uparrow},\simeq_{\mathsf{0}}\}}{\vdash}\overbrace{1+\cdots+1}^{n\text{ times}}+\overbrace{\mathsf{0}+\cdots+\mathsf{0}}^{k\text{ times}}\overset{\mathsf{DDI}^{\underline{\downarrow}}\cup\{\mathsf{i}^{\uparrow}\}}{\vdash}\overbrace{A+\cdots+A}^{n\text{ times}}\overset{\{+^{\downarrow},\mathsf{0}^{\downarrow},\simeq_{\mathsf{0}}\}}{\vdash} A$, where $m = n+k$ and $\simeq_{\mathsf{0}}$ is the restriction of the rule $\simeq$ to the equivalence $A+\mathsf{0}\simeq A$. By applying the following permutations

$$\begin{array}{c}\simeq_{\mathsf{0}}\dfrac{\mathsf{n}}{\boxed{\begin{array}{c}\mathsf{n}\\ \Vert\,\mathsf{DDI}^{\underline{\downarrow}}\cup\{\mathsf{i}^{\uparrow}\}\\ A+\cdots+A\end{array}}+\boxed{\mathsf{0}^{\downarrow}\dfrac{\mathsf{0}}{A}}}\\ \mathcal{D}\Vert\,\{+^{\downarrow}\}\\ A\end{array} \rightsquigarrow \begin{array}{c}\boxed{\begin{array}{c}\mathsf{n}\\ \Vert\,\mathsf{DDI}^{\underline{\downarrow}}\cup\{\mathsf{i}^{\uparrow}\}\\ A+\cdots+A\end{array}}\\ \mathcal{D}\Vert\,\{+^{\downarrow}\}\\ A\end{array} \qquad \begin{array}{c}\boxed{\begin{array}{c}\mathsf{n}\\ \Vert\,\mathsf{DDI}^{\underline{\downarrow}}\cup\{\mathsf{i}^{\uparrow}\}\\ A+\cdots+A\end{array}}+\boxed{\mathsf{0}^{\uparrow}\dfrac{1}{\mathsf{0}^{\downarrow}\dfrac{\mathsf{0}}{A}}}\\ \mathcal{D}\Vert\,\{+^{\downarrow}\}\\ A\end{array} \rightsquigarrow \begin{array}{c}\boxed{\begin{array}{c}\mathsf{n}\\ \Vert\,\mathsf{DDI}^{\underline{\downarrow}}\cup\{\mathsf{i}^{\uparrow}\}\\ A+\cdots+A\end{array}}\\ \mathcal{D}\Vert\,\{+^{\downarrow}\}\\ A\end{array}$$

$$\begin{array}{c}\simeq_{\mathsf{0}}\dfrac{\boxed{\begin{array}{c}\mathsf{n}\\ \Vert\,\mathsf{DDI}^{\underline{\downarrow}}\cup\{\mathsf{i}^{\uparrow}\}\\ A+\cdots+A\end{array}}+\boxed{\mathsf{0}^{\uparrow}\dfrac{1}{\mathsf{0}}}}{A+\cdots+A}\\ \mathcal{D}\Vert\,\{+^{\downarrow}\}\\ A\end{array} \rightsquigarrow \begin{array}{c}\boxed{\begin{array}{c}\mathsf{n}\\ \Vert\,\mathsf{DDI}^{\underline{\downarrow}}\cup\{\mathsf{i}^{\uparrow}\}\\ A+\cdots+A\end{array}}\\ \mathcal{D}\Vert\,\{+^{\downarrow}\}\\ A\end{array}$$

we obtain a derivation of A of the shape (where $\mathsf{n} = 1+\cdots+1$)

$$\simeq \frac{\boxed{\begin{array}{l}\quad\mathsf{1}\\ \mathcal{D}_1 \,\Vert\, \mathsf{DDI}^{\downarrow}_{-}\cup\{\mathsf{i}^{\uparrow}\}\\ \quad A\end{array}} + \cdots + \boxed{\begin{array}{l}\quad\mathsf{1}\\ \mathcal{D}_n \,\Vert\, \mathsf{DDI}^{\downarrow}_{-}\cup\{\mathsf{i}^{\uparrow}\}\\ \quad A\end{array}}}{\begin{array}{c} A+\cdots+A \\ \Vert\,\{+^{\downarrow}\} \\ A\end{array}} \qquad \text{or} \qquad \mathsf{0}^{\downarrow}\,\frac{\mathsf{0}}{A}$$

In the first case, we conclude by taking only one derivation among $\mathcal{D}_1,\ldots,\mathcal{D}_n$. □

As a corollary and generalization of Proposition 3.4.3, we have the following:

Lemma A.1. *If $A = B_1 + \cdots + B_m \neq \mathsf{0}$ is a formula in additive normal form and $\mathsf{m} \overset{\mathsf{SDDI}}{\longmapsto} A$, then there are $C_1,\ldots,C_n \in \{B_1,\ldots,B_m\}$ such that*

- *if $0 < k < m$, then $\mathsf{n} \overset{\mathsf{DDI}^{\downarrow}_{-}\cup\{\mathsf{i}^{\uparrow}\}}{\longmapsto} B_1 + \cdots + B_k + C_1 + \cdots + C_n \overset{\{+^{\downarrow},\mathsf{0}^{\downarrow}\}}{\longmapsto} A$;*
- *if $k = m$, then $\mathsf{n} \overset{\mathsf{DDI}^{\downarrow}_{-}\cup\{\mathsf{i}^{\uparrow}\}}{\longmapsto} B_1 + \cdots + B_k + C_1 + \cdots + C_n \overset{\{+^{\downarrow}\}}{\longmapsto} A$.*

B Cut-elimination in DiLL_0 sequent calculus

Rewriting rules $\rightsquigarrow_{\mathsf{cut}}$ for cut elimination in DiLL_0 sequent calculus are defined in Figure 6. They are just the formulation in the sequent calculus formalism of the cut elimination steps defined and studied in [6, 17, 9] within the interaction nets formalism.

We represent there only the *key cases*, where the principal connectives in the cut formulas are dual (the pairs of dual connectives are $\otimes/\parr$, $\mathsf{1}/\bot$, $!/?$). The way DiLL_0 deals with the *commutative cases* is omitted since is analogous to usual sequent calculi. With these cut elimination steps it has been proved in [6, 17, 9] that the rule cut is admissible in DiLL_0 (and even in DiLL, i.e., the system DiLL_0 plus the usual MELL promotion rule), see Theorem 2.4.

Note that if $\pi \rightsquigarrow_{\mathsf{cut}} \pi'$ with π canonical then π' is not necessarily canonical (*e.g.* if in π a cut $?\mathsf{c}/!\mathsf{d}$ or $?\mathsf{d}/!\mathsf{w}$ is above another rule), but π' can be rewritten in a canonical form (see Fact 2.2 above).

We give an informal account of the cut elimination steps in Figure 6 for the key cases involving $!/?$. Roughly, they follow the "law of supply and demand" so as to be resource-sensitive: in each slice no duplication or erasure is allowed. The rules for ? ($?\mathsf{w}$, $?\mathsf{d}$, $?\mathsf{c}$) *ask for* a number of resources of type $!A$ (0, 1, and the sum of the numbers asked by its premises, respectively), while the rules for ! ($!\mathsf{w}$, $!\mathsf{d}$, $!\mathsf{c}$) *supply* a number of resources of type $!A$ (0, 1, and the sum of the numbers supplied by its premises, respectively). There are several cases:

1. If the numbers of demanded and supplied resources match, the cut elimination proceeds normally (see the steps $?\mathsf{d}/!\mathsf{d}$ and $?\mathsf{w}/!\mathsf{w}$).
2. The step $?\mathsf{c}/!\mathsf{c}$ is slightly more complicated: it essentially connects the dual premises of a ?-contraction and of a !-contraction in all possible ways.
3. The step $?\mathsf{c}/!\mathsf{w}$ duplicates the rule $!\mathsf{w}$, spreading the information that there are no available resources to the premises of $?\mathsf{c}$.
4. The step $?\mathsf{d}/!\mathsf{w}$ represents a mismatch in supply and demand: ?-dereliction asks for a resource but !-weakening says that it is not available; the resulting derivation with the rule zero keeps track of this mismatch, as a sort of error in computation.

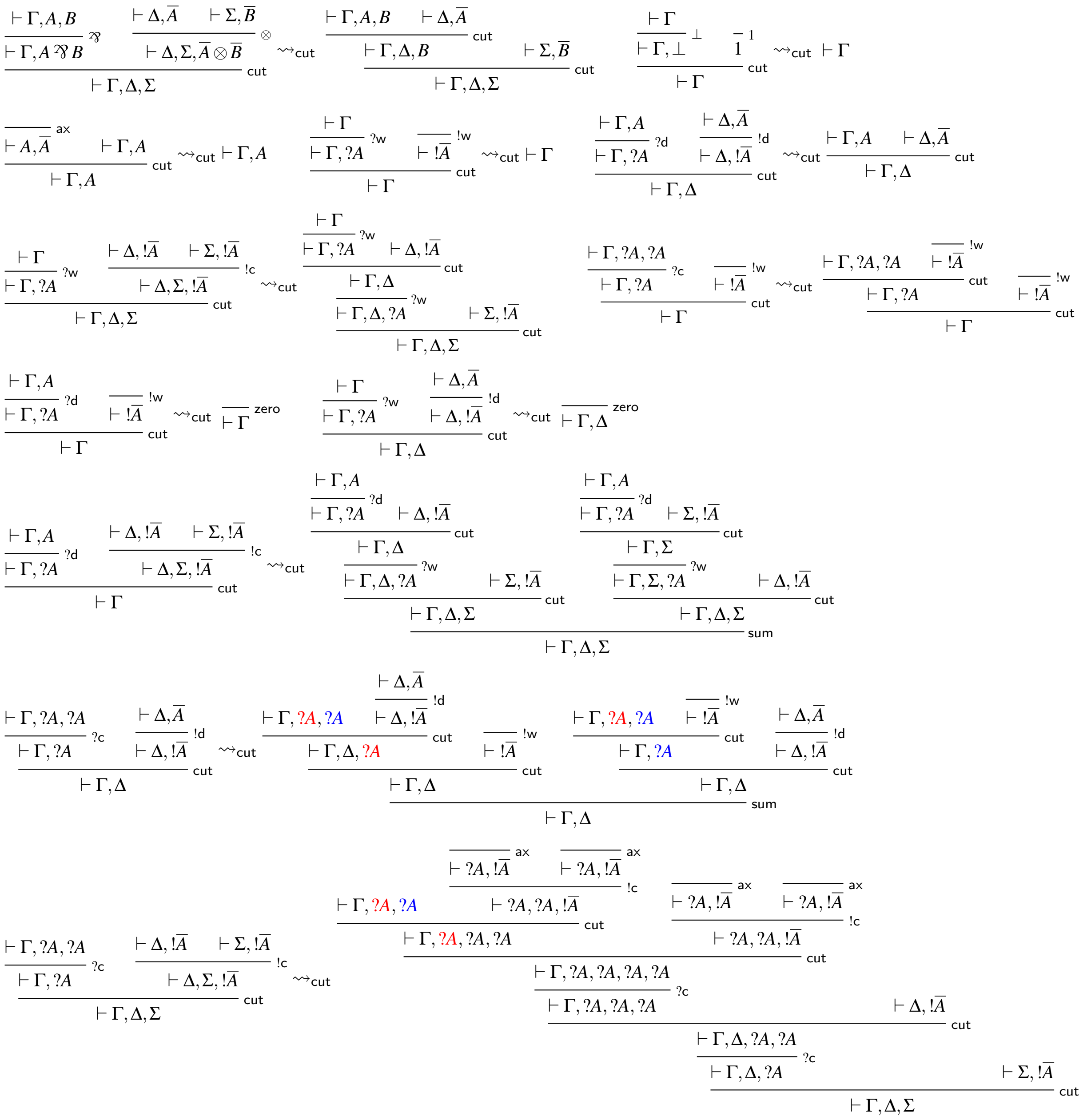

Figure 6: Cut elimination rewriting rules for DiLL_0 sequent calculus (key cases).

5. In the step ?c/!d, ?-contraction says that there are two possible demands for a resource, but according to !-dereliction only one resource is available, therefore there is a non-deterministic choice on which request will be fed, the other one will receive a !-weakening; the rule sum has to be intended as a way to keep track of all possible choices, not as a way to duplicate resources; said differently, in the step ?c/!d a derivation reduces to a pair of derivations (of slices, if we consider their canonical forms).

By duality, the discussion is similar for the steps ?w/!c, ?w/!d and ?d/!c, respectively.