AI504 Knowledge Representation
Exercises 3

Matteo Acclavio & Marco Santamaria

1. Let Σ be a signature containing a constant c, two function symbols f and g of arity 2, a predicate symbol P of arity 1 and a predicate symbols Q and E of arity 2.

   Consider the Σ-models M with domain the set of strictly positive natural number, and the following interpretation:

   - $c^M = 1$,
   - $f^M(n, m)$ is the greatest common divisor of n and m,
   - $g^M(n, m)$ is the least common multiple of n and m;
   - $P^M(n)$ is the subset of prime numbers;
   - $S^M(n, m)$ is the relation 'n divides m'.
   - $E^M(n, m)$ is the relation 'n is equal to m'.

   Say if the following formulas are valid or not in M:

   - $\forall x.\forall y.(E(x, y) \leftrightarrow E(y, x))$;
   - $\forall x.\forall y.S(f(x, y), g(x, y))$;
   - $\forall x.\forall y.(S(g(x, y), f(x, y)) \rightarrow E(x, y))$;
   - $\forall x.\forall y.(S(x, y) \rightarrow (P(x) \leftrightarrow P(y)))$;
   - $\forall x.\forall y.((S(x, y) \land P(y) \land \neg E(x, y)) \rightarrow E(x, c))$;
   - $\forall x.\forall y.((P(x) \land P(y)) \rightarrow (P(f(x, y)) \lor P(g(x, y))))$;
   - $\forall x.(\forall y.E(g(x, y), g(y, x)) \rightarrow E(x, c))$;
   - $\forall x.\exists y.(P(x) \rightarrow (\neg P(y) \land S(x, y)))$;
   - $\forall x.\exists y.\exists z.(\neg P(x) \rightarrow (\neg E(y, z) \land P(y) \land P(z) \land S(x, y) \land S(x, z)))$;

2. Let Σ be a signature with two predicates symbols $P, Q \in \mathcal{P}$ of arity 1. Prove if:

   - there are Σ-models M (with P and Q having different interpretations) such that:
     - $M \models \forall x.(P(x) \land Q(x)) \leftrightarrow (\forall x.P(x) \land \forall x.Q(x))$;
     - $M \models \forall x.(P(x) \lor Q(x)) \leftrightarrow (\forall x.P(x) \lor \forall x.Q(x))$;
     - $M \models \forall x.(P(x) \rightarrow Q(x)) \leftrightarrow (\forall x.P(x) \rightarrow \forall x.Q(x))$;
     - $M \models \forall x.(P(x) \rightarrow Q(x)) \rightarrow (\forall x.P(x) \rightarrow \exists x.Q(x))$;
     - $M \models \exists x.(P(x) \rightarrow Q(x)) \leftrightarrow (\forall x.P(x) \rightarrow \exists x.Q(x))$;
   - The following holds:
     - $\models \forall x.(P(x) \land Q(x)) \leftrightarrow (\forall x.P(x) \land \forall x.Q(x))$;
     - $\not\models \forall x.(P(x) \lor Q(x)) \leftrightarrow (\forall x.P(x) \lor \forall x.Q(x))$;
     - $\not\models \forall x.(P(x) \rightarrow Q(x)) \leftrightarrow (\forall x.P(x) \rightarrow \forall x.Q(x))$;
     - $\models \forall x.(P(x) \rightarrow Q(x)) \rightarrow (\forall x.P(x) \rightarrow \exists x.Q(x))$;
     - $\models \exists x.(P(x) \rightarrow Q(x)) \leftrightarrow (\forall x.P(x) \rightarrow \exists x.Q(x))$;

     Remark that the solutions to the first point (second and third equations) are not in contradiction with the second point!

3. Let Σ be a signature with a predicate symbol $R \in \mathcal{P}$ of arity 2. Prove or provide a counter-example of the following statements:

   - $M \models \exists y.\forall x.R(x,y) \to \forall x.\exists y.R(x,y)$.
   - if $M \models \forall x.\forall y.(R(x,y) \to R(y,x))$, and $M \models \forall x.\forall y.\forall z.((R(x,y) \land R(y,z)) \to R(x,z))$, then $M \models \forall x.R(x,x)$.
   - what if we also consider $M \models \forall x.\exists y.R(x,y)$ as a premise in the previous point?
   - if $M \models \forall x.\forall y.\forall z.((R(x,y) \land R(y,z)) \to R(x,z))$ and $M \models \forall x.\exists y.R(x,y)$, then $M \models \forall x.R(x,x)$.

4. Let Σ be a signature containing a predicate symbol P of arity 2, and consider the following formulas:

   (a) $A := \forall x.\exists y.P(x,y)$;

   (b) $B := \exists y.\forall x.P(x,y)$.

   Let M be a Σ-model with domain the set of natural numbers, where $P(x,y)$ is interpreted as 'x is greater or equal to y.

   - Translate the formulas A and B in sentences of natural language.
   - Is A satisfied by M?
   - Is B satisfied by M?
   - Is $A \to B$ valid in M? Does A logically entail B (in any model)?
   - Is $B \to A$ valid in M? Does B logically entail A (in any model)?

5. Consider the standard translation of modal formulas in the language of first-order logic:

   - for every propositional variable p, $\mathsf{ST}_x(p) = P(x)$ for a predicate symbol P of arity 1;
   - $\mathsf{ST}_x(\neg A) = \neg\mathsf{ST}_x(A)$;
   - $\mathsf{ST}_x(A \land B) = \mathsf{ST}_x(A) \land \mathsf{ST}_x(B)$
   - $\mathsf{ST}_x(A \lor B) = \mathsf{ST}_x(A) \lor \mathsf{ST}_x(B)$
   - $\mathsf{ST}_x(A \to B) = \mathsf{ST}_x(A) \to \mathsf{ST}_x(B)$;
   - $\mathsf{ST}_x(\Box A) = \forall y.(R(x,y) \to \mathsf{ST}_y(A))$;
   - $\mathsf{ST}_x(\Diamond A) = \exists y.(R(x,y) \land \mathsf{ST}_y(A))$;

   Let Σ be a signature containing only a predicate symbol of arity 2. Prove that each model satisfying the given first-order formula describing a condition on the frame is a model of the corresponding modal formula.

Frame condition	**First-Order Formula** (F)	**Modal Axiom** (AX)
Seriality	$\forall v.\exists w.R(v,w)$	$\mathsf{D} := \Box A \to \Diamond A$
Reflexivity	$\forall x.R(x,x)$	$\mathsf{T} := \Box A \to A$
Transitivity	$\forall u.\forall v.\forall w.\big((R(u,v) \land R(v,w)) \to R(u,w)\big)$	$4 := \Box A \to \Box\Box A$
Euclideanness	$\forall u.\forall v.\forall w.\big((R(u,v) \land R(u,w)) \to R(v,w)\big)$	$5 := \Diamond A \to \Box\Diamond A$
Symmetry	$\forall v.\forall w.\big(R(v,w) \to R(w,v)\big)$	$\mathsf{B} := A \to \Box\Diamond A$
Confluence	$\forall u.\forall v.\forall w.\big((R(u,v) \land R(u,w)) \to \exists u'.(R(v,u') \land R(w,u'))\big)$	$\mathsf{M} := \Diamond\Box A \to \Box\Diamond A$
Connectedness	$\forall v.\forall w.\big(R(v,w) \lor R(w,v)\big)$	$\mathsf{Dum} := \Box(\Box A \to B) \lor \Box(\Box B \to A)$
Density	$\forall u.\forall w.\big(R(u,w) \to \exists v.(R(u,v) \land R(v,w))\big)$	$\mathsf{Den} := \Box\Box A \to \Box A$

   To prove that also the converse holds, you have to consider the conjunction of the given axiom with the axiom $\mathsf{K} := \Box(A \to B) \to (\Box A \to \Box B)$. That is, $\mathsf{AX} \land \mathsf{K} \models F$

6. Compute, if it exists, the most general unifier of the following pairs of terms:

   (a) $f(x, g(y,z))$ and $f(g(a,b), g(y, h(y)))$;

   (b) $f(f(x), g(y))$ and $f(z, g(h(z)))$;

   (c) $h(x, f(y,z))$ and $h(f(a,b), f(y,c))$;

   (d) $g(x, f(y))$ and $g(f(a), f(b))$;

(e) $f(x, g(y, z))$ and $f(g(a, b), g(c, d))$;

7. Define a signature and use it to write down the following sentences using first-order formulas in such a way they are suitable for using Generalized Modus Ponens:

   - Horses, cows, and pigs are mammals.
   - An offspring of a horse is a horse.
   - Bluebeard is a horse.
   - Bluebeard is Charlie's parent.
   - Offspring and parent are inverse relations.
   - Every mammal has a parent.

   Then,

   - Draw the proof tree generated by an exhaustive backward-chaining algorithm for the query $\exists h.\mathsf{Horse}(h)$, where clauses are matched in the order given.
   - How many solutions for h actually follow from your sentences?
   - Can you think of an algorithm to find all of them?