AI504 Knowledge Representation
Exercises 1

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February 8, 2025

1. Which of the following are correct?

   (a) $(A \lor B) \land \neg(A \to B)$ is satisfiable.
   (b) $(A \leftrightarrow B) \land (\neg A \lor B)$ is satisfiable.
   (c) $\mathit{False} \models \mathit{True}$
   (d) $\mathit{True} \models \mathit{False}$
   (e) $A \land B \models A \leftrightarrow B$
   (f) $A \leftrightarrow B \models A \lor B$
   (g) $A \leftrightarrow B \models \neg A \lor B$
   (h) $(A \land B) \to C \models (A \to C) \lor (B \to C)$
   (i) $(A \lor B) \land (\neg C \lor \neg D \lor E) \models A \lor B$
   (j) $(A \lor B) \land (\neg C \lor \neg D \lor E) \models (A \lor B) \land (\neg D \lor E)$
   (k) $(C \lor (\neg A \land \neg B))$ is equivalent to $((A \to C) \land (B \to C))$
   (l) $(A \leftrightarrow B) \leftrightarrow C$ has the same number of models as $A \leftrightarrow B$ for any fixed set of proposition symbols that include A, B and C.

2. Prove each of the following assertions:

   (a) α is valid if and only if $\mathit{True} \models \alpha$.
   (b) For any α, $\mathit{False} \models \alpha$.
   (c) $\alpha \models \beta$ if and only if $\alpha \to \beta$ is valid.
   (d) $\alpha \equiv \beta$ if and only if $\alpha \leftrightarrow \beta$ is valid.
   (e) $\alpha \models \beta$ if and only if $\alpha \land \neg\beta$ is unsatisfiable.

3. Prove, or find a counterexample to, each of the following assertions:

   (a) If $\alpha \models \gamma$ or $\beta \models \gamma$ (or both) then $\alpha \land \beta \models \gamma$.
   (b) If $\alpha \land \beta \models \gamma$ then $\alpha \models \gamma$ or $\beta \models \gamma$ (or both).
   (c) If $\alpha \models \beta \lor \gamma$ then $\alpha \models \beta$ or $\alpha \models \gamma$ (or both).

4. Consider a vocabulary with only four propositions, A, B, C and D. How many models are there for the following sentences?

   (a) $B \lor C$
   (b) $\neg A \lor \neg B \lor \neg C \lor \neg D$
   (c) $(A \to B) \land A \land \neg B \land C \land D$.

5. Decide whether each of the following sentences is valid, unsatisfiable, or neither.

   (a) $\mathit{Smoke} \to \mathit{Smoke}$

(b) *Smoke* $\rightarrow$ *Fire*

(c) (*Smoke* $\rightarrow$ *Fire*) $\rightarrow$ ($\neg$*Smoke* $\rightarrow$ $\neg$*Fire*)

(d) *Smoke* $\lor$ *Fire* $\lor$ $\neg$*Fire*

(e) ((*Smoke* $\land$ *Heat*) $\rightarrow$ *Fire*) $\leftrightarrow$ ((*Smoke* $\rightarrow$ *Fire*) $\lor$ (*Heat* $\rightarrow$ *Fire*))

(f) (*Smoke* $\rightarrow$ *Fire*) $\rightarrow$ ((*Smoke* $\land$ *Heat*) $\rightarrow$ *Fire*)

(g) *Big* $\lor$ *Dumb* $\lor$ (*Big* $\rightarrow$ *Dumb*)

6. According to some political pundits, a person who is radical (R) is electable (E) if they are conservative (C), but otherwise is not electable.

   (a) Which of the following are correct representations of this assertion?

      i. $(R \land E) \leftrightarrow C$

      ii. $R \rightarrow (E \leftrightarrow C)$

      iii. $R \rightarrow ((C \rightarrow E) \lor \neg E)$

   (b) Which of the sentences in (a) can be expressed in Horn form?

7. Consider the following list of axiom schemas for propositional logic, where α, β, and γ are arbitrary sentences, and primitive connectives are $\neg$, $\land$, $\lor$, and $\rightarrow$.

   (A1) $\alpha \rightarrow (\beta \rightarrow \alpha)$

   (A2) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$

   (A3) $\alpha \land \beta \rightarrow \alpha$

   (A4) $\alpha \land \beta \rightarrow \beta$

   (A5) $\alpha \rightarrow (\beta \rightarrow (\alpha \land \beta))$

   (A6) $\alpha \rightarrow \alpha \lor \beta$

   (A7) $\beta \rightarrow \alpha \lor \beta$

   (A8) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \lor \beta \rightarrow \gamma))$

   (A9) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha)$

   (A10) $\neg\neg\alpha \rightarrow \alpha$

   It is possible to prove *completeness*: every tautology can be derived from the above set of axioms using *modus ponens* as the sole inference rule. In this inference system, the biconditional connective is introduced by definition: $\alpha \leftrightarrow \beta$ stands for $(\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)$. Remember that the *deduction theorem* holds: $\Gamma, \alpha \vdash \beta$ implies $\Gamma \vdash \alpha \rightarrow \beta$.

   (a) Using truth tables, prove that axioms A1-A10 are tautologies.

   (b) Prove that $\alpha \rightarrow \beta, \beta \rightarrow \gamma \vdash \alpha \rightarrow \gamma$

   (c) Using modus ponens and deduction theorem, prove the following tautologies.

      i. $\alpha \rightarrow \alpha$

      ii. $(\neg\beta \rightarrow \neg\alpha) \rightarrow ((\neg\beta \rightarrow \alpha) \rightarrow \beta)$

      iii. $(\neg\alpha \rightarrow \neg\beta) \leftrightarrow (\beta \rightarrow \alpha)$

      iv. $\neg(\alpha \lor \beta) \rightarrow (\neg\alpha \land \neg\beta)$

      v. $(\neg\alpha \lor \neg\beta) \rightarrow \neg(\alpha \land \beta)$

      vi. $\alpha \rightarrow \neg\neg\alpha$