Introduction to Proof Equivalence

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Class 4: Proof Equivalence in Classical Logic

matteoacclavio.com/Course.html?course=2023-ESSLLI

Classical Logic

Formulas

$$A, B := a \mid \bar{a} \mid A \wedge B \mid A \vee B$$

Sequent Calculus LK

$$\text{ax}\,\frac{}{a, \bar{a}} \quad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \quad \wedge\,\frac{\Gamma, A \quad B, \Delta}{\Gamma, A \wedge B, \Delta} \quad \text{w}\,\frac{\Gamma}{\Gamma, A} \quad \text{c}\,\frac{\Gamma, A, A}{\Gamma, A}$$

Theorem

LK *is a sound and complete proof system for classical logic.*

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Theorem

LK *is a sound and complete proof system for classical logic.*

Theorem

Cut elimination holds in LK.

Denotational Semantics for Classical Logic

$$\begin{array}{rccc} \{\!\{-\}\!\}: & \{\text{ Proofs }\} & \to & \{\text{ Denotations }\} \\ & \mathfrak{D} & \to & \{\!\{\mathfrak{D}\}\!\} \end{array}$$

such that:

- if $\mathfrak{D}$ proves $A \vdash B$ and $\mathfrak{D}'$ proves $B \vdash C$, then it is defined $\mathfrak{D} * \mathfrak{D}'$ proving $A \vdash C$;
- if $\mathfrak{D} \rightsquigarrow \mathfrak{D}'$ (via cut-elimination/normalization/...), then $\{\!\{\mathfrak{D}\}\!\} = \{\!\{\mathfrak{D}'\}\!\}$

Lafont's pair

$$
\mathrm{cut}\;\dfrac{\mathrm{w}\;\dfrac{\begin{array}{c}\mathfrak{D}_1 \Vert \\ A\end{array}}{\vdash A, B} \qquad \mathrm{w}\;\dfrac{\begin{array}{c}\mathfrak{D}_2 \Vert \\ A\end{array}}{\vdash \bar{B}, A}}{\mathrm{c}\;\dfrac{\vdash A, A}{\vdash A}}
$$

Joyal's argoment: any denotational semantics for LK is trivial
(any cartesian closed category with an initial object 0 such that $0^{0^A} \simeq A$ is a poset)

We cannot have a denotational semantics for LK

We cannot have a denotational semantics for LK

Or can we? . . .

A possible solution: reduce the proof space

Polarized formulas

$$A, B := a \mid \bar{a} \mid A \wedge^- B \mid A \vee^- B \mid A \wedge^+ B \mid A \vee^+ B$$

Focused sequent calculus LKF:

Asynchronous rules

$$\wedge^- \frac{\vdash A, \Gamma \Uparrow \Theta \quad \vdash B, \Gamma \Uparrow \Theta}{\vdash A \wedge^- B, \Gamma \Uparrow \Theta} \qquad \vee^- \frac{\vdash A, B, \Gamma \Uparrow \Theta}{\vdash A \vee^- B, \Gamma \Uparrow \Theta}$$

Synchronous rules

$$\wedge^+ \frac{\vdash A \Downarrow \Theta \quad \vdash B \Downarrow \Theta}{\vdash A \wedge^+ B \Downarrow \Theta} \qquad \vee^+ \frac{\vdash A_i \Gamma \Downarrow \Theta}{\vdash A_1 \vee^+ A_2, \Gamma \Downarrow \Theta}$$

Initial, store, release, and decide rules

$$\text{init} \frac{}{\vdash a \Downarrow \bar{a}, \Theta} \qquad \text{store} \frac{\vdash \Gamma \Uparrow Q, \Theta}{\vdash \Gamma, Q \Uparrow \Theta} \qquad \text{release} \frac{\vdash N \Uparrow \Theta}{\vdash N \Downarrow \Theta} \qquad \text{decide} \frac{\vdash P \Uparrow P, \Theta}{\vdash \cdot \Uparrow P, \Theta}$$

where N is a negative formula, P is positive,
Q is a positive formula or a negative atom

Theorem

Let Γ be a sequent, Θ a set of formulas, A a formula. Then

- $\vdash \Gamma \Uparrow \Theta$ *is provable in* LKF *iff* $\vdash \hat{\Gamma}, \hat{\Theta}$ *is provable in* LK
- $\vdash A \Downarrow \Theta$ *is provable in* LKF *iff* $\vdash \hat{A}, \hat{\Theta}$ *is provable in* LK

Theorem (Cut-elimination)

The following cut-*rules are eliminable in* LKF*:*

$$\mathrm{cut}_u\ \frac{\vdash A, \Gamma \Uparrow \Theta \quad \vdash \bar{A}, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'} \qquad \mathrm{cut}_u\ \frac{\vdash A \Downarrow \Theta \quad \vdash \bar{A}, \Gamma' \Uparrow \Theta'}{\vdash \Gamma' \Uparrow \Theta, \Theta'}$$

$$\mathrm{dcut}_f\ \frac{\vdash \Gamma \Uparrow \Theta, P \quad \vdash \bar{P}, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'} \qquad \mathrm{dcut}_f\ \frac{B \Downarrow \Theta, P \quad \bar{P} \Uparrow \Theta'}{\vdash B \Downarrow \Theta, \Theta'}$$

!!! In LKF we cannot have the Lafont's pair !!!

Combinatorial Proofs

(back to generality!)

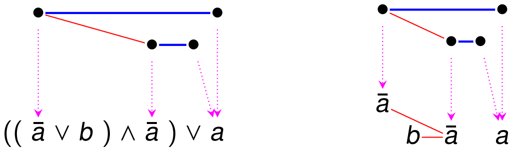
Combinatorial Proofs

Definition

A combinatorial proof of a formula F is an axiom-preserving skew fibration

$$f : \mathcal{G} \to [\![F]\!]$$

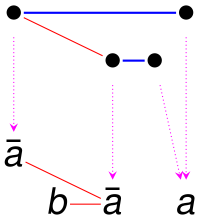
from a **RB**-cograph $\mathcal{G}$ to the cograph of F.



Idea:

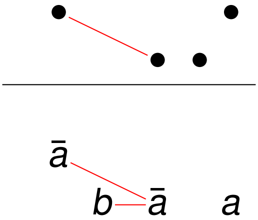
- cograph = graph enconding a formula
- **RB**-cograph = MLL proof nets
- skew fibration = $\{W^{\downarrow}, C^{\downarrow}\}$-derivations (ALL proof nets)

Cographs[1]



[1] Duffin 1965

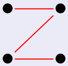
Cographs[1]



[1] Duffin 1965

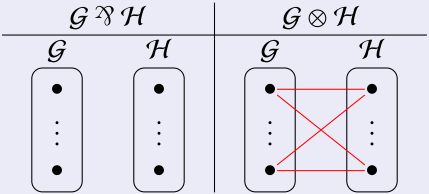
Cographs

Definition

A **cograph** is a graph containing no four vertices such that 

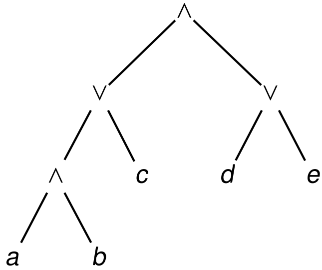
Theorem

A graph is a cograph iff constructed from single-vertices graphs using the graph operations



From formula to cographs

$$((a \wedge b) \vee c) \wedge (d \vee e)$$



a	b	
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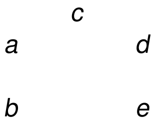

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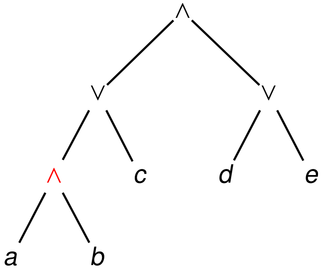

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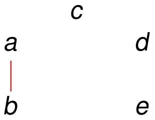

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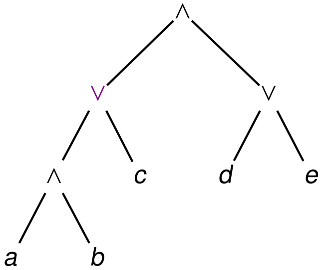

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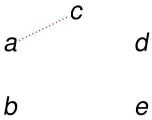

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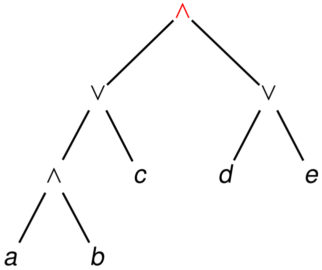

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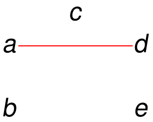

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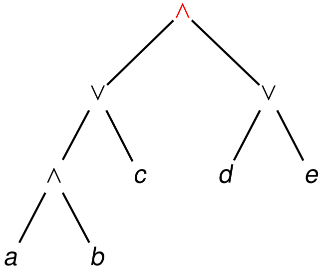

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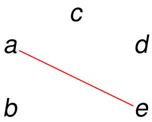

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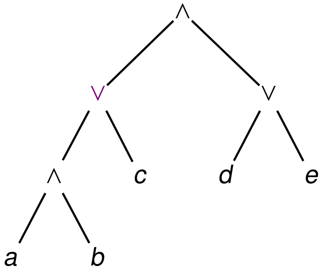

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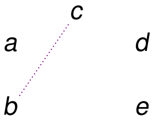

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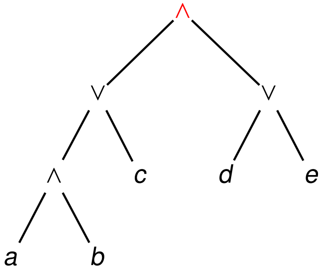

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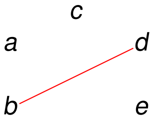

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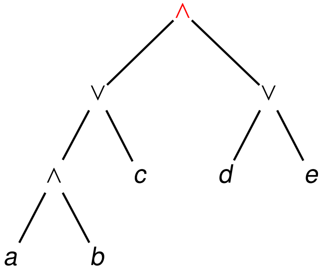

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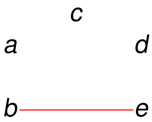

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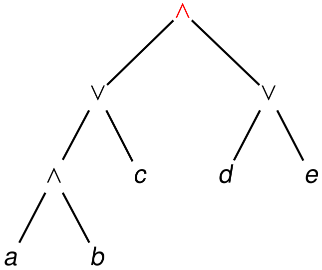

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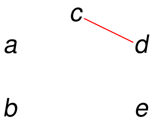

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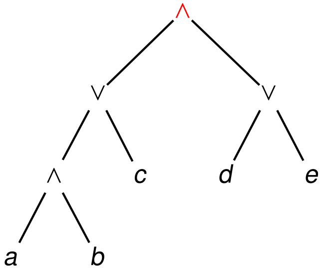

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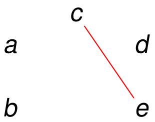

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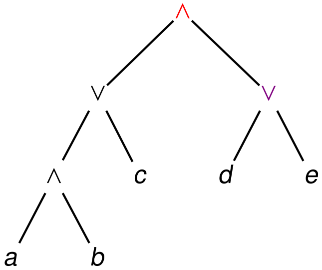

a	b	
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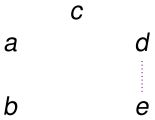

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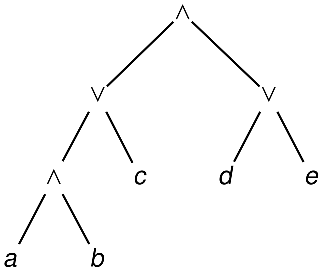

a	b	$\frown$
a	c	$\not\frown$
a	d	$\frown$
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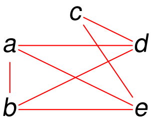
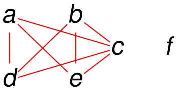
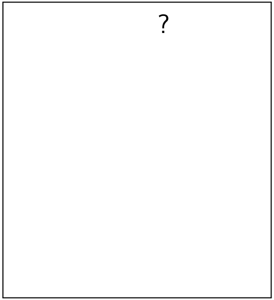


From cographs to formulas

Lemma

If $\mathcal{G}$ is a cograph, then either $\mathcal{G}$ or $\bar{\mathcal{G}}$ is disconnected.





Formula = ?

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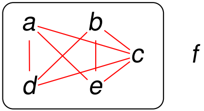

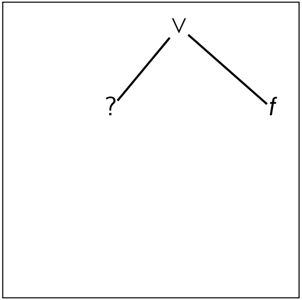

Formula = ? ∨ *f*

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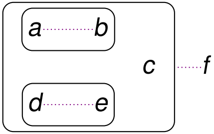

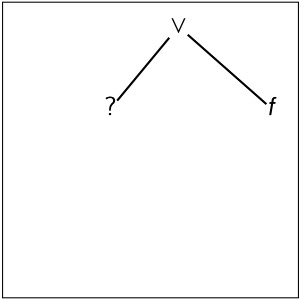

Formula = ? $\vee$ f

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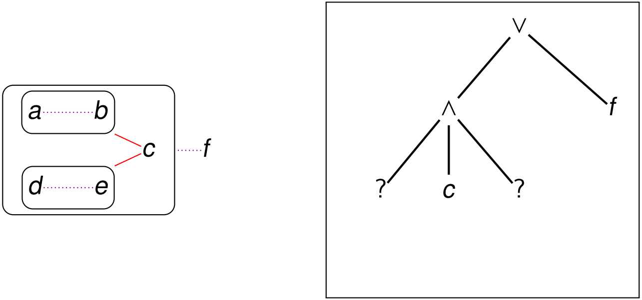

Formula = $(? \wedge c \wedge ?) \vee f$

From cographs to formulas

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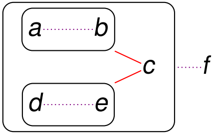
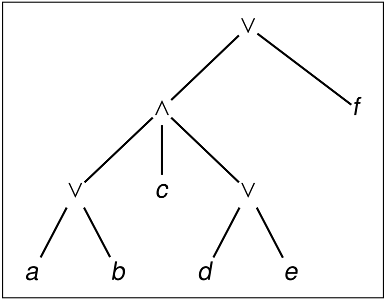




Formula = $((a \vee b) \wedge c \wedge (d \vee e)) \vee f$

Cograph and Formula Isomophism

Definition

The formula isomorphism $\simeq$ is the equivalence relation generated by:

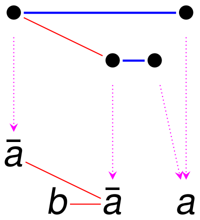
$$A \wedge B \simeq B \wedge A \qquad A \vee B \simeq B \vee A$$

$$(A \wedge B) \wedge C \simeq A \wedge (B \wedge C) \qquad (A \vee B) \vee C \simeq A \vee (B \vee C)$$

Theorem

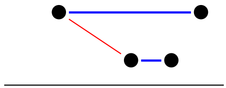
$$F \simeq F' \iff [\![F]\!] = [\![F']\!]$$

RB-cographs[2]



[2] Retoré 1993

RB-cographs[2]



[2] Retoré 1993

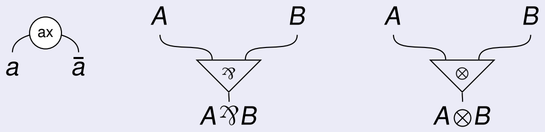
MLL Proof nets

The sequent calculus for LK

$$\text{ax}\,\frac{}{a, \bar{a}} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \wedge\,\frac{\Gamma, A \quad B, \Delta}{\Gamma, A \wedge B, \Delta} \qquad \frac{\Gamma}{\Gamma, A}\,\text{W} \qquad \frac{\Gamma, A, A}{\Gamma, A}\,\text{C}$$

Definition

A **proof structure** is a graph constructed using the following links



A **proof net** is a proof structure encoding a derivation in MLL

MLL Proof nets

The sequent calculus for MLL

$$\text{ax}\,\frac{}{a, \bar{a}} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \wedge\,\frac{\Gamma, A \quad B, \Delta}{\Gamma, A \wedge B, \Delta}$$

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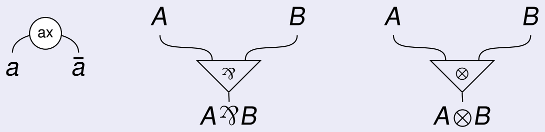

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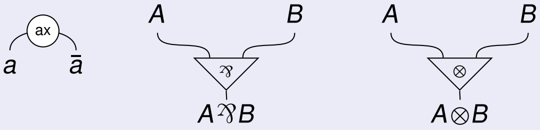
MLL Proof nets

The sequent calculus for MLL

$$\text{ax}\ \frac{}{a, \bar{a}} \qquad ⅋\ \frac{\Gamma, A, B}{\Gamma, A ⅋ B} \qquad \otimes\ \frac{\Gamma, A \quad B, \Delta}{\Gamma, A \otimes B, \Delta}$$

Definition

A **proof structure** is a graph constructed using the following links

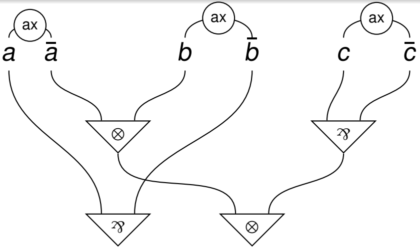


A **proof net** is a proof structure encoding a derivation in MLL

MLL Proof nets

Definition

A proof structure is correct if "pruning" one input from each $\parr$-gate we obtain a connected and acyclic graph.



Definition

A proof net is correct iff it is connected and acyclic (for each **switching**).

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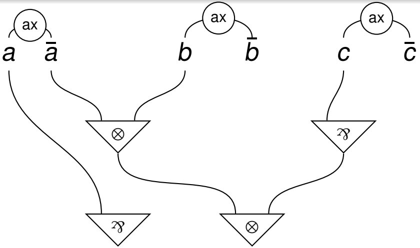

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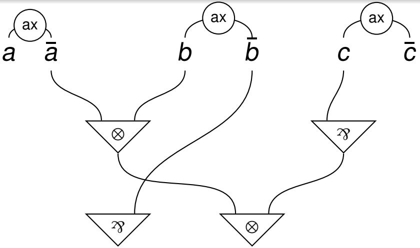

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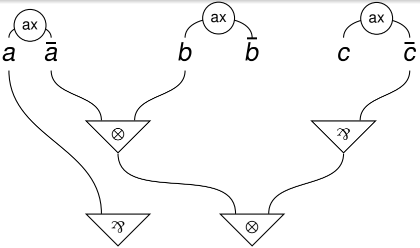

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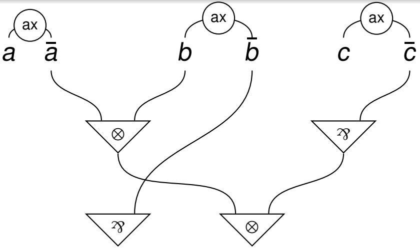

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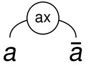
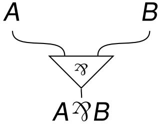
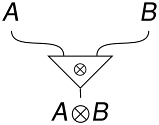
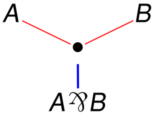
Handsome proof nets

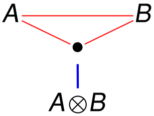
$\text{ax}\ \dfrac{}{a, \bar{a}}$	$⅋\ \dfrac{\Gamma, A, B}{\Gamma, A ⅋ B}$	$\otimes\ \dfrac{\Gamma, A \quad B, \Delta}{\Gamma, A \otimes B, \Delta}$











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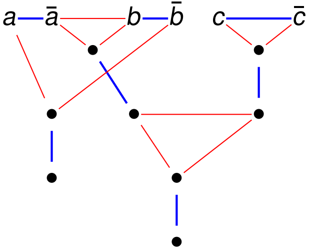
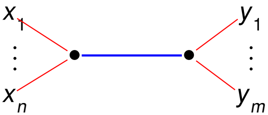


Definition

A **RB**-proof net is correct iff it is æ-connected and æ-acyclic.

Handsome proof nets: unfolding

Unfolding = remove •-vertices from the graph



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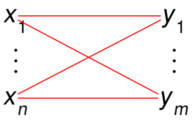

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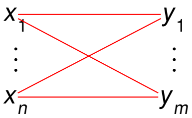

Note: by removing •-vertices we remove all non-axiom $\rightharpoonup$-edges

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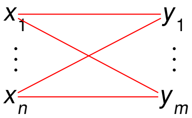

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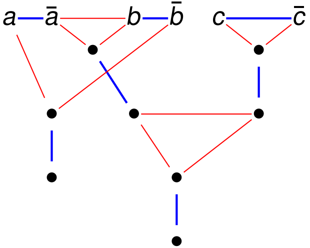

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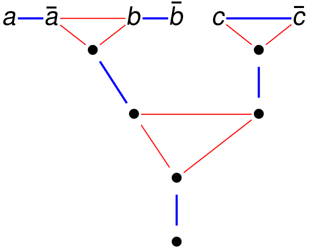
A **RB**-cograph is correct iff it is æ-connected and æ-acyclic .

Handsome proof nets: unfolding



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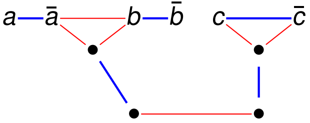

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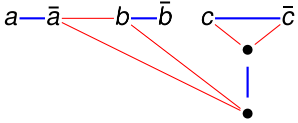

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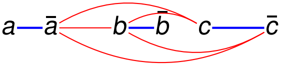

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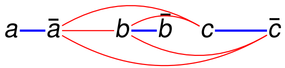
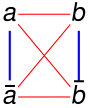
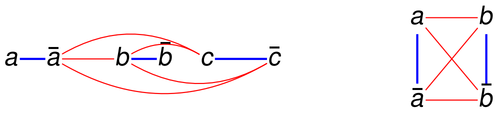




Definition

A **RB**-cograph is correct iff it is æ-connected and æ-acyclic w.r.t. **cordless paths**.

RB-cograph



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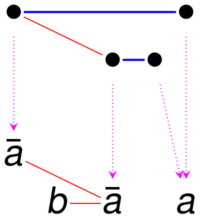
Theorem

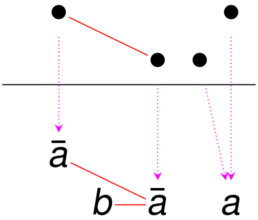
$\overset{\text{MLL}}{\vdash} F \iff$ *exists a correct* ***RB****-cograph* $\langle V, \frown, \rightharpoonup \rangle$ *s.t.* $[\![F]\!] = \langle V, \frown \rangle$

Skew Fibrations[3]



[3] Hughes 2005; Straßburger RTA2007

Skew Fibrations[3]



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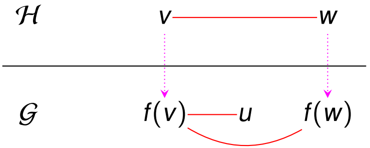

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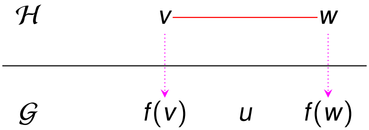

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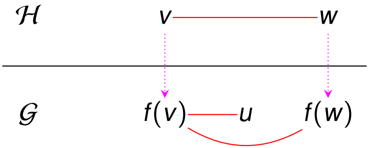

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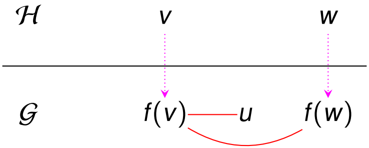

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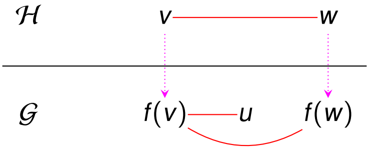

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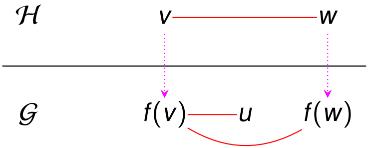

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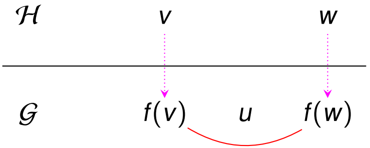

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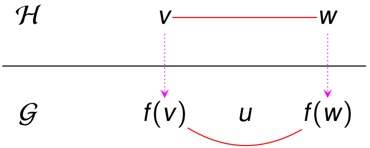

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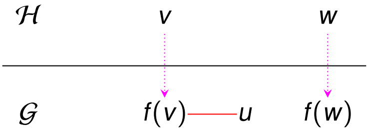

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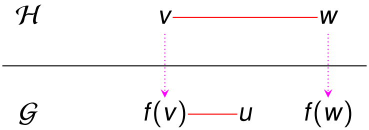

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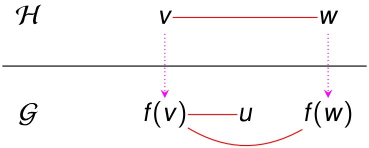

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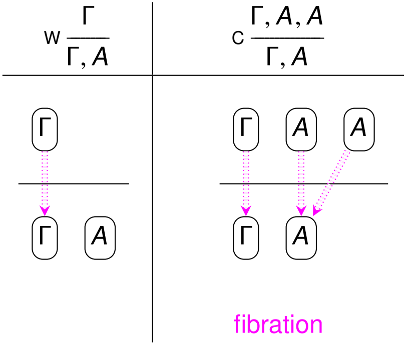
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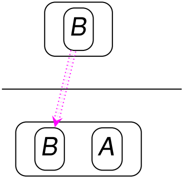
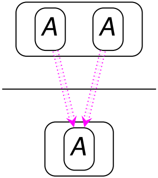
fibration

$$\mathsf{w}{\downarrow}\ \frac{\mathcal{C}\{B\}}{\mathcal{C}\{B \vee A\}} \qquad \mathsf{c}{\downarrow}\ \frac{\mathcal{C}\{A \vee A\}}{\mathcal{C}\{A\}}$$





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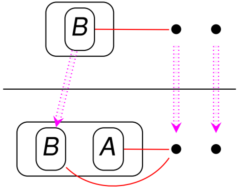

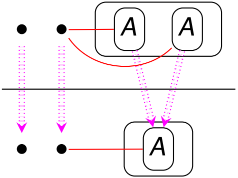

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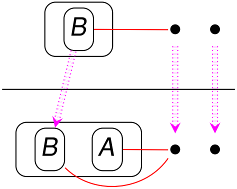
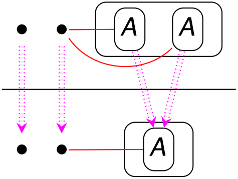
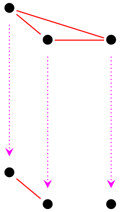


skew



fibration

Skew Fibrations (midterm exam)



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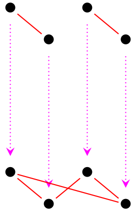

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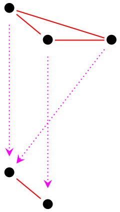
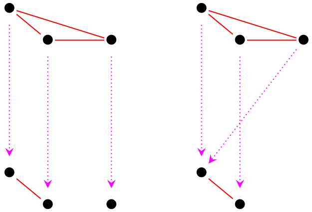
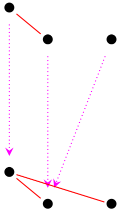


Skew Fibrations (midterm exam)

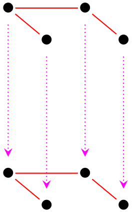


Skew Fibrations (midterm exam)



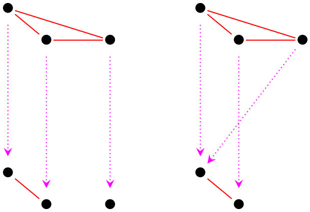
Skew Fibrations (midterm exam)

Skew Fibrations (midterm exam)

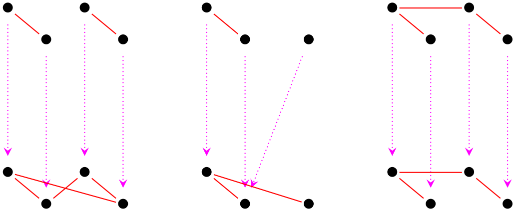


Skew Fibrations (midterm exam)

Is a not skew fibration



Is a skew fibration



Skew Fibrations[4]

Theorem (Decomposition)

$$F' \overset{\{\mathsf{w}^{\downarrow},\mathsf{c}^{\downarrow}\}}{\longmapsto} F \Longrightarrow \textit{there is a skew fibration } f : [\![F']\!] \to [\![F]\!]$$

$$\begin{array}{c} F' \\ \Big\| \{\mathsf{w}^{\downarrow},\mathsf{c}^{\downarrow}\} \\ F \end{array} \quad \Longrightarrow \quad \begin{array}{c} F' \\ \Big\| \mathsf{m}^{\downarrow} \\ G' \\ \Big\| \mathsf{ac}^{\downarrow} \\ G \\ \Big\| \mathsf{w}^{\downarrow} \\ F \end{array} \quad \Longrightarrow \quad \begin{array}{c} [\![F']\!] \\ \downarrow \text{bijective} \\ [\![G']\!] \\ \downarrow \text{surjective} \\ [\![G]\!] \\ \downarrow \text{injecive} \\ [\![F]\!] \end{array}$$

$$\mathsf{m}^{\downarrow}\ \frac{C\{(A \wedge C) \vee (B \wedge D)\}}{C\{(A \vee B) \wedge (C \vee D)\}}$$

$$\mathsf{ac}^{\downarrow}\ \frac{C\{a \vee a\}}{C\{a\}}$$

$$\mathsf{w}^{\downarrow}\ \frac{C\{B\}}{C\{A \vee B\}}$$

[4] Hughes 2005 ; Straßburger RTA2007

Reassembling the pieces

Combinatorial Proofs

What we have:

- **RB**-cograph: a graphical syntax for MLL proofs
- Skew fibrations: graph homomorphisms representing $\{W^{\downarrow}, C^{\downarrow}\}$-derivations

What do we what:

- Combine them to have a graphical syntax for $\mathrm{LK} = \mathrm{MLL} \cup \{W, C\}$

$$ {\scriptstyle\wedge}\dfrac{ {\scriptstyle \mathsf{C}}\dfrac{\begin{array}{c}\Vert\\ \Gamma, A, A\end{array}}{\Gamma, A} \qquad {\scriptstyle\vee}\dfrac{{\scriptstyle \mathsf{W}}\dfrac{\begin{array}{c}\Vert\\ \Delta, B\end{array}}{\Delta, B, C}}{\Delta, B\vee C}}{\Gamma,\Delta, A\wedge(B\vee C)} \qquad \rightsquigarrow ? $$

Combinatorial Proofs

What we have:

- **RB**-cograph: a graphical syntax for MLL proofs
- Skew fibrations: graph homomorphisms representing $\{\mathsf{W}^{\downarrow}, \mathsf{C}^{\downarrow}\}$-derivations

What do we what:

- Combine them to have a graphical syntax for $\mathsf{LK} = \mathsf{MLL} \cup \{\mathsf{W}, \mathsf{C}\}$

$$\wedge\,\dfrac{\mathsf{C}\,\dfrac{\overset{\Vert}{\Gamma, A, A}}{\Gamma, A} \qquad \vee\,\dfrac{\mathsf{W}\,\dfrac{\overset{\Vert}{\Delta, B}}{\Delta, B, C}}{\Delta, B \vee C}}{\Gamma, \Delta, A \wedge (B \vee C)} \quad \rightsquigarrow \quad \mathsf{C}^{\downarrow}\,\dfrac{\mathsf{W}^{\downarrow}\,\dfrac{\wedge\,\dfrac{\vee\,\dfrac{\overset{\Vert}{\Gamma, A, A}}{\Gamma, A \vee A} \qquad \overset{\Vert}{\Delta, B}}{\Gamma, \Delta, (A \vee A) \wedge B}}{\Gamma, \Delta, (A \vee A) \wedge (B \vee C)}}{\Gamma, \Delta, A \wedge (B \vee C)}$$

Combinatorial Proofs

Theorem (Decomposition)

$$\overset{\mathsf{LK}}{\vdash} F \Longrightarrow \overset{\mathsf{MLL}}{\vdash} F' \overset{\{\mathsf{W}^{\downarrow},\mathsf{C}^{\downarrow}\}}{\vdash} F$$

$$\begin{array}{c} \| \mathsf{LK} \\ F \end{array}$$

Theorem

Every LK *derivation can be represented by a combinatorial proof*

Combinatorial Proofs

Theorem (Decomposition)

$$\overset{\mathsf{LK}}{\longmapsto} F \implies \overset{\mathsf{MLL}}{\longmapsto} F' \overset{\{\mathsf{w}^{\downarrow},\mathsf{c}^{\downarrow}\}}{\longmapsto} F$$

$$\begin{array}{c} \| \mathsf{LK} \\ F \end{array} \quad\implies\quad \begin{array}{c} \mathfrak{D}' \| \mathsf{MLL} \\ F' \\ \mathfrak{D} \| \{\mathsf{w}^{\downarrow},\mathsf{c}^{\downarrow}\} \\ F \end{array}$$

Theorem

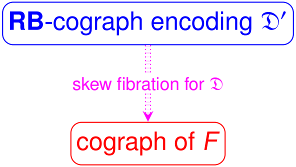
Every LK *derivation can be represented by a combinatorial proof*

Combinatorial Proofs

Theorem (Decomposition)

$$\vdash^{\mathsf{LK}} F \implies \vdash^{\mathsf{MLL}} F' \vdash^{\{\mathsf{w}^{\downarrow},\mathsf{c}^{\downarrow}\}} F$$

$$\begin{array}{c} \| \text{LK} \\ F \end{array} \implies \begin{array}{c} \mathfrak{D}' \| \text{MLL} \\ F' \\ \mathfrak{D} \| \{\mathsf{w}^{\downarrow},\mathsf{c}^{\downarrow}\} \\ F \end{array} \implies$$



Theorem

Every LK *derivation can be represented by a combinatorial proof*

Combinatorial Proofs

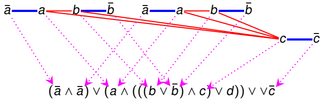
Theorem

Every combinatorial proof can be sequentialized into a derivation in $\mathsf{LK} \cup \{\mathsf{cut}\}$

What is the problem with the converse?
Hughes's example:

$$
\mathsf{w}{\downarrow}\,\dfrac{\mathsf{asso}\,\dfrac{\wedge\,\dfrac{\mathsf{C}\,\dfrac{\wedge\,\dfrac{\wedge\,\dfrac{\mathsf{ax}\,\dfrac{}{a,\bar a}\quad \vee\,\dfrac{\mathsf{ax}\,\dfrac{}{b,\bar b}}{b\vee\bar b}}{a\wedge(b\vee\bar b),\bar a}\qquad \wedge\,\dfrac{\mathsf{ax}\,\dfrac{}{a,\bar a}\quad \vee\,\dfrac{\mathsf{ax}\,\dfrac{}{b,\bar b}}{b\vee\bar b}}{a\wedge(b\vee\bar b),\bar a,}}{a\wedge(b\vee\bar b),a\wedge(b\vee\bar b),\bar a\wedge\bar a,}}{a\wedge(b\vee\bar b),\bar a\wedge\bar a}\qquad \mathsf{ax}\,\dfrac{}{c,\bar c}}{(a\wedge(b\vee\bar b))\wedge c,\bar a\wedge\bar a,\bar c}}{a\wedge((b\vee\bar b)\wedge c),\bar a\wedge\bar a,\bar c}}{a\wedge(((b\vee\bar b)\wedge c)\vee d),\bar a\wedge\bar a,\bar c}
$$

$\Longleftrightarrow$



Theorem

$$F' \overset{\{\mathsf{w}\downarrow,\mathsf{c}\downarrow,\equiv\}}{\vdash} F \iff \textit{there is a skew fibration } f \colon [\![F']\!] \to [\![F]\!]$$

Combinatorial Proofs form a Proof System

Fact (Cook-Reckhow)

Check whether a syntactic object represents a valid proof can be done by means of a polynomial time algorithm.

- Check if a graph is a cograph
- Check if a **RB**-cograph is æ-connected and æ-acyclic
- Check if a map $f : \mathcal{H} \to \mathcal{G}$ between cograph is a skew fibration
- Check if f is axiom-preserving

Theorem

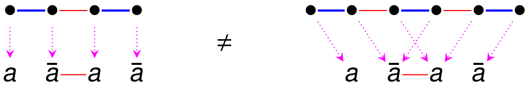
Combinatorial Proofs form a proof system for classical logic.

Combinatorial Proofs and Proof Equivalence

Combinatorial Proofs and Proof equivalence

Claim

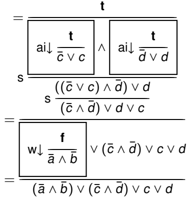
Two proofs are the same iff they can be represented by the same CP

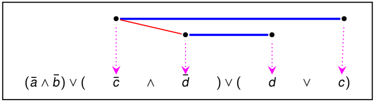


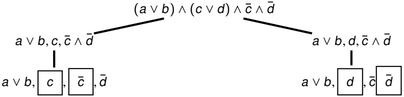
- Combinatorial Proofs and sequent calculus
- Combinatorial Proofs and deep inference
- Combinatorial Proofs and Resolution and Analytic Tableaux

Comparing Proofs from Different Proof Systems

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bar c, c}\,\mathsf{AX}}{\vdash \bar c, c, d}\,\mathsf{W}}{\vdash (\bar a \wedge \bar b), \bar c, c, d}\,\mathsf{W} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \bar d, d}\,\mathsf{AX}}{\vdash \bar d, c, d}\,\mathsf{W}}{\vdash (\bar a \wedge \bar b), \bar d, c, d}\,\mathsf{W}}{\vdash (\bar a \wedge \bar b), (\bar c \wedge \bar d), c, d}\,\wedge}{\vdash (\bar a \wedge \bar b), (\bar c \wedge \bar d), c, d}\,\vee}{\vdash (\bar a \wedge \bar b) \vee (\bar c \wedge \bar d), c, d}\,\vee}{\vdash (\bar a \wedge \bar b) \vee (\bar c \wedge \bar d) \vee c, d}\,\vee}{\vdash (\bar a \wedge \bar b) \vee (\bar c \wedge \bar d) \vee c \vee d}\,\vee}{}
$$







$$
\dfrac{\dfrac{\dfrac{[(a \vee b) \wedge (c \vee d) \wedge \bar c \wedge \bar d]}{[a \vee b][(c \vee d) \wedge \bar c \wedge \bar d]}\,\wedge}{[a \vee b][c \vee d][\bar c \wedge \bar d]}\,\wedge}{[a \vee b][\,]}\,Res^{c \vee d}
$$

Proof Equivalence in Sequent Calculus

Independent rule permutations

$$\dfrac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\,\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\,\rho_1 \equiv \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\,\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\,\rho_2$$

$$\dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\,\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\,\rho_2 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\,\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\,\rho_1 \qquad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\,\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\,\rho_1 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\,\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\,\rho_2$$

Comonoid transformations

$$\dfrac{\dfrac{\Gamma,A_1,A_2,A_3}{\Gamma,A_1,A}\,\mathsf{C}}{\Gamma,A}\,\mathsf{C} \simeq \dfrac{\dfrac{\Gamma,A_1,A_2,A_3}{\Gamma,A,A_3}\,\mathsf{C}}{\Gamma,A}\,\mathsf{C} \qquad \dfrac{\dfrac{\Gamma,A,A}{\Gamma,A}\,\mathsf{C}}{\Gamma,A,A}\,\mathsf{W} \simeq \Gamma,A,A \qquad \dfrac{\dfrac{\Gamma,A}{\Gamma,A,A}\,\mathsf{W}}{\Gamma,A}\,\mathsf{C} \simeq \Gamma,A$$

Permutations of structural rules with comma/∨

$$\dfrac{\dfrac{\Gamma,A,B,A,B}{\Gamma,A\vee B,A\vee B}\,2\times\vee}{\Gamma,A\vee B}\,\mathsf{C} \simeq \dfrac{\dfrac{\Gamma,A,B,A,B}{\Gamma,A,B}\,2\times\mathsf{C}}{\Gamma,A\vee B}\,\vee \qquad \dfrac{\dfrac{\Gamma}{\Gamma,A,B}\,2\times\mathsf{W}}{\Gamma,A\vee B}\,\vee \simeq \dfrac{\Gamma,A,B,A,B}{\Gamma,A\vee B}\,\mathsf{W}$$

Exchising and Unfolding

$$\wedge\,\dfrac{\begin{array}{c}\pi\Vert\\ \Gamma,A\end{array} \quad \mathsf{W}\,\dfrac{\begin{array}{c}\pi'\Vert\\ \Delta\end{array}}{B,\Delta}}{\Gamma,A\wedge B,\Delta} \simeq \mathsf{W}\,\dfrac{\begin{array}{c}\pi'\Vert\\ \Delta\end{array}}{\Gamma,A\wedge B,\Delta}$$

$$\wedge\,\dfrac{\begin{array}{c}\pi\Vert\\ \Gamma,A\end{array} \quad \mathsf{C}\,\dfrac{\begin{array}{c}\pi'\Vert\\ B,B,\Delta\end{array}}{B,\Delta}}{\Gamma,A\wedge B,\Delta} \simeq \mathsf{C}\,\dfrac{\wedge\,\dfrac{\begin{array}{c}\pi\Vert\\ \Gamma,A\end{array} \quad \wedge\,\dfrac{\begin{array}{c}\pi\Vert\\ \Gamma,A\end{array} \quad \begin{array}{c}\pi'\Vert\\ B,B,\Delta\end{array}}{\Gamma,A\wedge B,B,\Delta}}{\Gamma,A\wedge B,\Gamma,A\wedge B,\Delta}}{\Gamma,A\wedge B,\Delta}$$

Compositionality

How to represent cut

Combinatorial proofs allows to represent cut-free proofs

$\rightsquigarrow$ $\rightsquigarrow$

$\rightsquigarrow$ $\rightsquigarrow$

How to represent cut

Combinatorial proofs allows to represent cut-free proofs

Fact

Proof of Γ *with a cut on a formula* $A \iff$ *Proof of* $\Gamma, A \wedge \bar{A}$

$\rightsquigarrow$ $\rightsquigarrow$

$\rightsquigarrow$ $\rightsquigarrow$

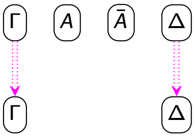
How to represent cut

Combinatorial proofs allows to represent cut-free proofs

Fact

Proof of Γ *with a cut on a formula* A $\iff$ *Proof of* $\Gamma, A \wedge \bar{A}$

$$\text{cut}\,\frac{\Gamma, A \quad \bar{A}, \Delta}{\Gamma, \Delta} \qquad \rightsquigarrow \qquad\qquad \rightsquigarrow$$



$\rightsquigarrow \qquad\qquad \rightsquigarrow$

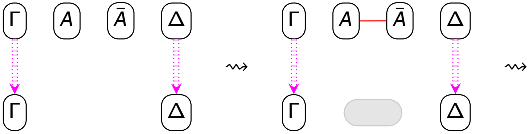
How to represent cut

Combinatorial proofs allows to represent cut-free proofs

Fact

Proof of Γ *with a cut on a formula* $A \iff$ *Proof of* $\Gamma, A \wedge \bar{A}$

$$\text{cut}\,\frac{\Gamma, A \quad \bar{A}, \Delta}{\Gamma, \Delta} \quad \rightsquigarrow \quad \text{hide}\,\frac{\wedge\,\dfrac{\Gamma, A \quad \bar{A}, \Delta}{\Gamma, {\color{red}A \wedge \bar{A}}, \Delta}}{\Gamma, \Delta} \quad \rightsquigarrow$$



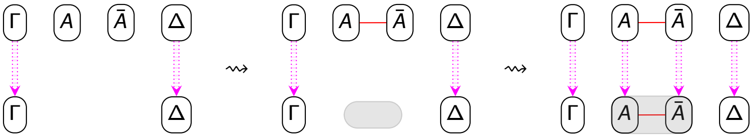
How to represent cut

Combinatorial proofs allows to represent cut-free proofs

Fact

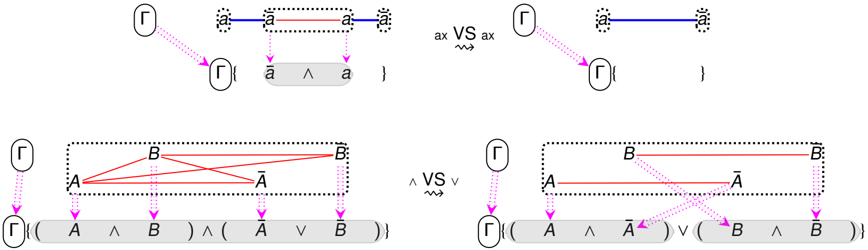
Proof of Γ with a cut on a formula A $\iff$ Proof of $\Gamma, A \wedge \bar{A}$

$$\text{cut}\,\frac{\Gamma, A \quad \bar{A}, \Delta}{\Gamma, \Delta} \quad \rightsquigarrow \quad \text{hide}\,\frac{\wedge\,\dfrac{\Gamma, A \quad \bar{A}, \Delta}{\Gamma, A \wedge \bar{A}, \Delta}}{\Gamma, \Delta} \quad \rightsquigarrow \quad \wedge\,\frac{\Gamma, A \quad \bar{A}, \Delta}{\Gamma, A \wedge \bar{A}, \Delta}$$



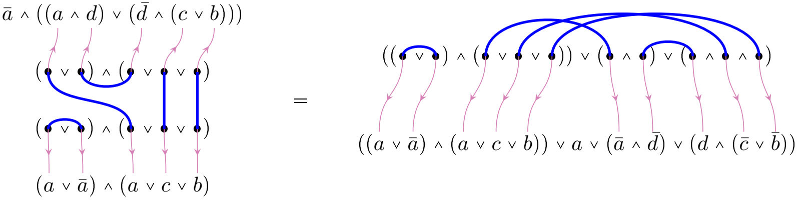
Cut-elimination

Cut-elimination = elimination of contradictions



Cut-elimination (a different approach)

A different approach:



The (current) realm of Combinatorial Proofs

CPs for Relevant and Affine Logics

- Relevant Logic = LK without weakening
- Affine Logic = LK without contraction

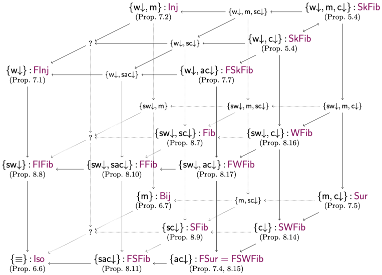


*figure from Ralph and Straßburger Tablueaux2019 paper

- Entailment Logic (non associative connectives)

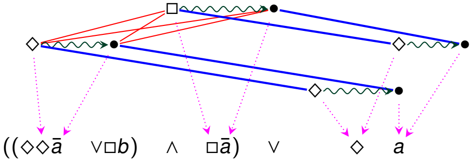
Modal Logic S4

Modal Formulas

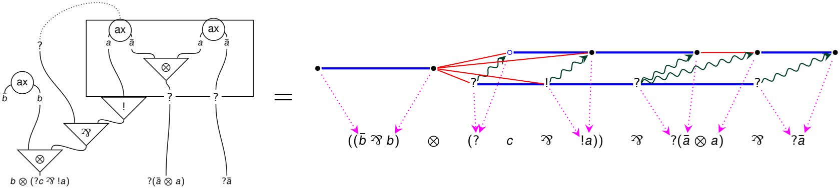
$$A, B := a \mid \bar{a} \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A$$

Sequent Calculus Rules

$$\mathsf{LK} \cup \left\{ \mathsf{K}\ \frac{A, \Gamma}{\Box A, \Diamond \Gamma} \quad , \quad \mathsf{D}\ \frac{A, \Gamma}{\Diamond A, \Diamond \Gamma} \quad , \quad \mathsf{T}^{\downarrow}\ \frac{\mathcal{C}\{A\}}{\mathcal{C}\{\Diamond A\}} \quad , \quad 4^{\downarrow}\ \frac{\mathcal{C}\{\Diamond\Diamond A\}}{\mathcal{C}\{\Diamond A\}} \right\}$$



Multiplicative Linear Logic with Exponentials

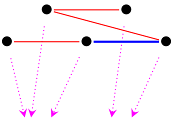


First Order Classical Logic

Formulas

$$\begin{aligned} t &:= c \mid f(t_1, \ldots, t_n) \\ a &:= p(t_1, \ldots, t_n) \mid \bar{p}(t_1, \ldots, t_n) \\ A, B &:= a \mid A \wedge B \mid A \vee B \mid \forall x A \mid \exists x A \end{aligned}$$

$$\text{Rules LK} \cup \left\{ \exists \frac{\Gamma, A[x/t]}{\Gamma, \exists x.A} \quad , \quad \forall \frac{\Gamma, A}{\Gamma, \forall x.A}\ x \text{ not free in } \Gamma \right\}$$



$\exists x \bar{p}(x) \forall y p(y)$

Intuitionistic Logic

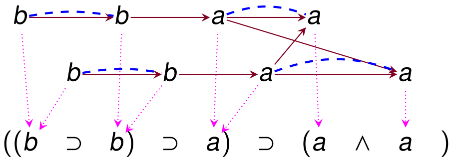
Formulas

$$A, B \coloneqq a \mid A \wedge B \mid A \supset B$$

Sequent Calculus Rules

$$\frac{}{a \vdash a}\,\text{ax} \qquad \frac{\Gamma, B \vdash A}{\Gamma \vdash B \supset A}\supset^{\text{R}} \qquad \frac{\Gamma, B, C \vdash A}{\Gamma, B \wedge C \vdash A}\wedge^{\text{L}} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\wedge^{\text{R}} \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\supset^{\text{L}}$$

$$\frac{}{\vdash 1}\,1 \qquad \frac{\Gamma, B, B \vdash A}{\Gamma, B \vdash A}\,\text{C} \qquad \frac{\Gamma \vdash A}{\Gamma, B \vdash A}\,\text{W}$$



Proof Equivalence in LJ

Definition

The proof equivalence in

Natural Deduction = λ-calculus = Winning Innocent Strategies

is given by

Rules permutations + Comonoid transformations + Unfolding + Excising

Definition

The proof equivalence in

Intuitionistic Combinatorial Proofs

is given by

Rules permutations + Comonoid transformations + Exchising

Independent rules

$$\frac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 \equiv \frac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$$

$$\frac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 \equiv \frac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1 \qquad \frac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 \equiv \frac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$$

Resource Management

$$\frac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A,B\vdash C}2\times C}{\Gamma,A\wedge B\vdash C}\wedge^L \equiv_c \frac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}2\times\wedge^L}{\Gamma,A\wedge B\vdash C}C \qquad \frac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}2\times W}{\Gamma,A\wedge B\vdash C}\wedge^L \equiv_c \frac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}W$$

$$\frac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}C}{\Gamma,A,A\vdash B}W \equiv_c \Gamma,A,A\vdash B \qquad \frac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}W}{\Gamma,A\vdash B}C \equiv_c \Gamma,A\vdash B$$

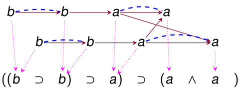
Excising and Unfolding

$$\frac{\Gamma\vdash A \quad \dfrac{\Delta\vdash C}{B,\Delta\vdash C}W}{\Gamma,\Delta,A\supset B\vdash C}\supset^L \equiv_e \frac{\Delta\vdash C}{\Gamma,\Delta,A\supset B\vdash C}W \qquad \frac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}C}{\Gamma,A\supset B\vdash C}\supset^L \equiv_u \frac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\supset^L}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\supset^L}{\Gamma,\Delta,A\supset B\vdash C}C$$

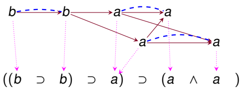
$$\equiv_{CP} := (\equiv \cup \equiv_c \cup \equiv_e) \qquad \equiv_{WIS} := (\equiv \cup \equiv_c \cup \equiv_e \cup \equiv_u)$$

Proof Equivalence in LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\,\text{AX}}{\vdash b \supset b}\,\supset^R \quad \dfrac{}{a \vdash a}\,\text{AX}}{(b \supset b) \supset a \vdash a}\,\supset^L \quad \dfrac{\dfrac{\dfrac{}{b \vdash b}\,\text{AX}}{\vdash b \supset b}\,\supset^R \quad \dfrac{}{a \vdash a}\,\text{AX}}{(b \supset b) \supset a \vdash a}\,\supset^L}{(b \supset b) \supset a, (b \supset b) \supset a \vdash a \wedge a}\,\wedge^R}{(b \supset b) \supset a \vdash a \wedge a}\,\text{C}$$



$$\dfrac{\dfrac{\dfrac{}{b \vdash b}\,\text{AX}}{\vdash b \supset b}\,\supset^R \quad \dfrac{\dfrac{\dfrac{}{a \vdash a}\,\text{AX} \quad \dfrac{}{a \vdash a}\,\text{AX}}{a, a \vdash a \wedge a}\,\wedge^L}{a \vdash a \wedge a}\,\text{C}}{(b \supset b) \supset a \vdash a \wedge a}\,\supset^L \qquad \dfrac{\dfrac{\dfrac{}{b \vdash b}\,\text{AX}}{\vdash b \supset b}\,\supset^R \quad \dfrac{\dfrac{\dfrac{}{a \vdash a}\,\text{AX} \quad \dfrac{}{a \vdash a}\,\text{AX}}{a, a \vdash a \wedge a}\,\wedge^L}{a \vdash a \wedge a}\,\text{C}}{(b \supset b) \supset a \vdash a \wedge a}\,\supset^L$$



Both these proofs correspond to the same WIS

$$\left\{ \epsilon, \begin{array}{c} a_0 \;,\; a_0 a \;,\; a_0 ab \;,\; a_0 abb \\ \\ a_2 \;,\; a_2 a \;,\; a_2 ab \;,\; a_2 abb \end{array} \right\}$$

Are two proofs using different amounts of the same resources equal?

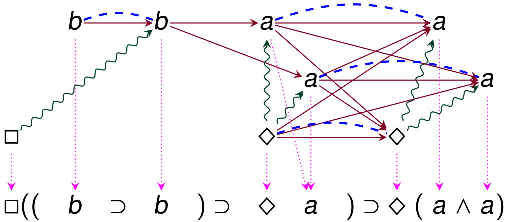
Constructive Modal Logic

Modal Formulas

$$A, B \coloneqq a \mid A \wedge B \mid A \supset B \mid \Box A \mid \Diamond A \mid 1$$

Additional Sequent Calculus Rules

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\mathsf{K}_{\Box} \qquad \frac{B, \Gamma \vdash A}{\Diamond B, \Box\Gamma \vdash \Diamond A}\,\mathsf{K}_{\Diamond} \qquad \frac{B, \Gamma \vdash A}{\Box\Gamma \vdash \Diamond A}\,\mathsf{D}$$



Independent rules	$\dfrac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$ $\quad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1$ $\quad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$
Resource Management	$\dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A,B\vdash C}2\times C}{\Gamma,A\wedge B\vdash C}\wedge^L \equiv_c \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}2\times\wedge^L}{\Gamma,A\wedge B\vdash C}C$ $\quad \dfrac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}2\times W}{\Gamma,A\wedge B\vdash C}\wedge^L \equiv_c \dfrac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}W$ $\dfrac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}C}{\Gamma,A,A\vdash B}W \equiv_c \Gamma,A,A\vdash B$ $\quad \dfrac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}W}{\Gamma,A\vdash B}C \equiv_c \Gamma,A\vdash B$
Excising and Unfolding	$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta\vdash C}{B,\Delta\vdash C}W}{\Gamma,\Delta,A\supset B\vdash C}\supset^L \equiv_e \dfrac{\Delta\vdash C}{\Gamma,\Delta,A\supset B\vdash C}W$ $\quad \dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}C}{\Gamma,A\supset B\vdash C}\supset^L \equiv_u \dfrac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\supset^L}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\supset^L}{\Gamma,\Delta,A\supset B\vdash C}C$
Structural vs K	$\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}W}{\Box\Gamma,\Box B\vdash\Box A}K_\Box \equiv_{\Box c} \dfrac{\dfrac{\Gamma\vdash A}{\Box\Gamma\vdash\Box A}K_\Box}{\Box\Gamma,\Box B\vdash\Box A}W$ $\quad \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Gamma,B\vdash A}C}{\Box\Gamma,\Box B\vdash\Box A}K_\Box \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Box\Gamma,\Box B,\Box B\vdash\Box A}K_\Box}{\Box\Gamma,\Box B\vdash\Box A}C$ $\dfrac{\dfrac{\Gamma,B\vdash A}{\Gamma,B,C\vdash A}W}{\Box\Gamma,\Diamond B,\Box C,\vdash\Box A}K_\Diamond \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B\vdash A}{\Box\Gamma,\Diamond B\vdash\Diamond A}K_\Diamond}{\Box\Gamma,\Diamond B,\Box C\vdash\Box A}W$ $\quad \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Gamma,B,C\vdash A}C}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}K_\Box \equiv_{\Box c} \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Box\Gamma,\Diamond B,\Box C,\Box C\Box\vdash\Diamond A}K_\Box}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}C$
Jumps	$\dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}W}{\Box\Gamma,\Diamond B\vdash\Diamond A}K_\Diamond}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}W \equiv_{\Diamond w} \dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,C\vdash A}W}{\Box\Gamma,\Diamond C\vdash\Diamond A}K_\Diamond}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}W$

$\equiv_{CP} := (\equiv \cup \equiv_c \cup \equiv_e) \qquad \equiv_\lambda := (\equiv_{CP} \cup \equiv_u) \qquad \equiv_{WIS} := (\equiv_\lambda \cup \equiv_{\Box c}) \qquad \equiv_{\Diamond w} := (\equiv_{WIS} \cup \equiv_{\Box c})$