ESSLLI 2023
> LJUBLJANA > SLOVENIA

Introduction to Proof Equivalence

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Class 3.5 (a.k.a. 6): Denotational Semantics and Game Semantics

matteoacclavio.com/Course.html?course=2023-ESSLLI

Denotational Semantics

$$
\begin{array}{rccc}
\{\!\{-\}\!\}: & \{\ \text{Proofs}\ \} & \to & \{\ \text{Denotations}\ \} \\
 & \mathfrak{D} & \to & \{\!\{\mathfrak{D}\}\!\}
\end{array}
$$

such that:

- if $\mathfrak{D}$ proves $A \vdash B$ and $\mathfrak{D}'$ proves $B \vdash C$, then it is defined $\mathfrak{D} * \mathfrak{D}'$ proving $A \vdash C$;
- if $\mathfrak{D} \rightsquigarrow \mathfrak{D}'$ (via cut-elimination/normalization/...), then $\{\!\{\mathfrak{D}\}\!\} = \{\!\{\mathfrak{D}'\}\!\}$

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Lazy solution: $\dfrac{\{\text{derivations}\}}{\text{cut-elimination}}$

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Lazy solution: $\dfrac{\{\text{derivations}\}}{\text{cut-elimination}}$

Categorical semantics: $\mathcal{C}$ with $\mathcal{O}_{\mathcal{C}} = \{\text{Formulas}\}$ and $\mathcal{O}_{\mathcal{C}} = \{\text{Proofs of } A \vdash B\}$

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Categorical semantics: $\mathcal{C}$ with $\mathcal{O}_{\mathcal{C}} = \{\text{Formulas}\}$ and $\mathcal{O}_{\mathcal{C}} = \{\text{Proofs of } A \vdash B\}$

Let's try to have something more concrete!

Negative fragment of intuitionisitc logic

Formulas

$$A, B ::= 1 \mid a \mid A \supset B \mid A \wedge B$$

Sequent Calulus

$$\frac{}{a \vdash a}\,\mathrm{AX} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\supset^{\mathrm{R}} \quad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\supset^{\mathrm{L}} \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\wedge^{\mathrm{R}} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\wedge^{\mathrm{L}}$$

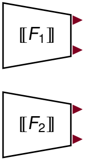
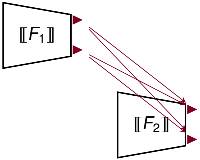
$$\frac{}{\vdash 1}\,1 \quad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathrm{C} \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathrm{W} \qquad \boxed{\frac{\Gamma, \vdash A \quad A, \Delta, \vdash C}{\Gamma, \Delta \vdash C}\,\mathrm{cut}}$$

Game Semantics for Intuitionistic Logic

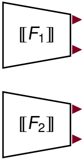
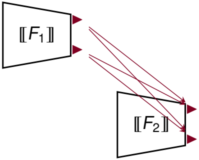
Game Semantics name is ambiguous:

- Lorenzen and Lorenz, Felscher, Stubborn, Fermüller . . . : proof search as dialogue
- Blass, Conway, Abramsky, Hyland, Ong, . . . : denotational semantics for proofs

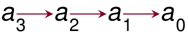
Arena of a formula

$[\![1]\!]$	$[\![a]\!]$	$[\![F_1 \wedge F_2]\!]$	$[\![F_1 \supset F_2]\!]$
$\emptyset$	a		
empty graph	single vertex with label a	the disjoint union of $[\![F_1]\!]$ and $[\![F_2]\!]$	the disjoint union of $[\![F_1]\!]$ and $[\![F_2]\!]$ plus $\rightarrow$-edges from any vertex in $[\![F_1]\!]$ with no outgoing $\rightarrow$-edge to any vertex in $[\![F_2]\!]$ with no outgoing $\rightarrow$-edge

Arena of a formula

$[\![1]\!]$	$[\![a]\!]$	$[\![F_1 \wedge F_2]\!]$	$[\![F_1 \supset F_2]\!]$
$\emptyset$	a		
empty graph	single vertex with label a	the disjoint union of $[\![F_1]\!]$ and $[\![F_2]\!]$	the disjoint union of $[\![F_1]\!]$ and $[\![F_2]\!]$ plus $\rightarrow$-edges from any vertex in $[\![F_1]\!]$ with no outgoing $\rightarrow$-edge to any vertex in $[\![F_2]\!]$ with no outgoing $\rightarrow$-edge

Examples:

$[\![((a_3 \supset a_2) \supset a_1) \supset a_0]\!] =$ 

$[\![((b_1 \supset b_0) \supset a_1) \supset (a_2 \wedge a_0)]\!] =$ 

$[\![((a_2 \wedge a_0) \supset b_1) \supset (a_1 \supset b_0)]\!] =$ 

Rules of the game for a formula F:

- The "board" is the arena $[\![F]\!]$
- Two-players game ($\circ$ and $\bullet$)
- The player $\circ$ starts on a root
- Each player (non-initial) move is *justified* by one previous opponent move[1]
- The player $\bullet$ must "reply" to the previous $\circ$-move
- a player wins when the other is out of moves

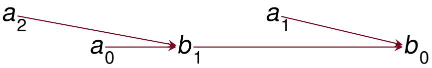
Definition (WIS)

- **Play**: sequence of moves
- **Winning strategy**: set of plays considering every possible $\circ$-move
- **Innocent**: each $\circ$-move is justified by the previous $\bullet$-move ($\circ$ is *shortsighted*)

[1] We ask that the node of this move points ($\rightarrow$) the node of a justifying move

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

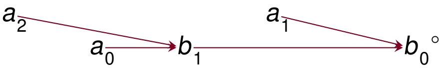
It is $\circ$'s turn



$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ \\ \\ \\ \end{array} \right\}$$

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

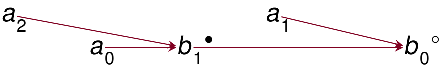
It is •'s turn



$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^\circ \\ \\ \\ \\ \end{array} \right\}$$

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

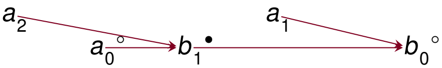
It is $\circ$'s turn



$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} \\ \\ \\ \\ \end{array} \right\}$$

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

It is $\bullet$'s turn



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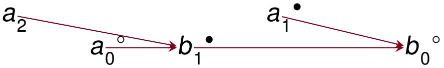
Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

It is ∘'s turn

PLAYER • WINS!



$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^\circ \\ b_0^\circ b_1^\bullet \\ b_0^\circ b_1^\bullet a_0^\circ \\ b_0^\circ b_1^\bullet a_0^\circ a_1^\bullet \\ \\ \\ \end{array} \right\}$$

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

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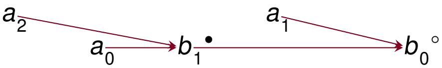
$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} \\ b_0^{\circ} b_1^{\bullet} a_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} a_0^{\circ} a_1^{\bullet} \\ \\ \end{array} \right\}$$

Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

It is $\bullet$'s turn



$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} \\ b_0^{\circ} b_1^{\bullet} a_0^{\circ} \\ b_0^{\circ} b_1^{\bullet} a_0^{\circ} a_1^{\bullet} \\ \\ \\ \end{array} \right\}$$

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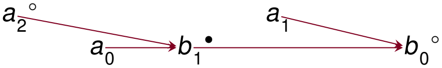

$$\mathcal{S} = \left\{ \begin{array}{l} \epsilon \\ b_0^\circ \\ b_0^\circ b_1^\bullet \\ b_0^\circ b_1^\bullet a_0^\circ \\ b_0^\circ b_1^\bullet a_0^\circ a_1^\bullet \\ b_0^\circ b_1^\bullet a_2^\circ \\ \end{array} \right\}$$

Let's play on $[\![((a \land a) \supset b) \supset (a \supset b)]\!]$

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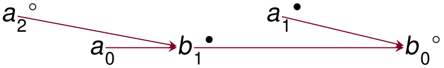

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Let's play on $[\![((a \wedge a) \supset b) \supset (a \supset b)]\!]$

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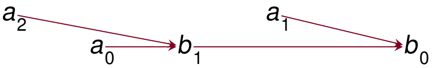

We can write the set of maximal views in $\mathcal{S}$ as follows

$$\mathcal{M}ax(\mathcal{S}) = \left\{ \begin{array}{l} b_0 b_1 a_0 a_1 \\ b_0 b_1 a_2 a_1 \end{array} \right\}$$

where dotted lines identify the justifier of a move

Let's play on $[\![((a \supset a) \supset a) \supset a]\!]$

It is $\circ$'s turn

$$a_3 \longrightarrow a_2 \longrightarrow a_1 \longrightarrow a_0^{\circ}$$

$$\mathcal{M}ax(\mathcal{S}_1) = \{a_0 \qquad \}$$

$$\mathcal{M}ax(\mathcal{S}_2) = \{ \qquad \}$$

$$\mathcal{M}ax(\mathcal{S}_3) = \{ \qquad \}$$

!We here writing the maximal views of three distinct WISs!
(dotted lines identify the justifier of a move)

Let’s play on $[\![((a \supset a) \supset a) \supset a]\!]$

It is $\bullet$’s turn

$$a_3 \longrightarrow a_2 \longrightarrow a_1 \overset{\bullet}{\longrightarrow} a_0$$

$$\mathcal{M}ax(\mathcal{S}_1) = \{ a_0 a_1 \}$$

$$\mathcal{M}ax(\mathcal{S}_2) = \{ \}$$

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$$a_3 \longrightarrow a_2 \overset{\bullet}{\longrightarrow} a_1 \longrightarrow a_0$$

$$\mathcal{M}ax(\mathcal{S}_1) = \{a_0 a_1 a_2 \quad\}$$

$$\mathcal{M}ax(\mathcal{S}_2) = \{ \quad \}$$

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Let's play on $[\![((a \supset a) \supset a) \supset a]\!]$

It is $\bullet$'s turn

PLAYER $\bullet$ WINS!

$$a_3 \overset{\bullet}{\longrightarrow} a_2 \longrightarrow a_1 \longrightarrow a_0$$

$$\mathcal{M}ax(\mathcal{S}_1) = \{a_0 a_1 a_2 a_3\}$$

$$\mathcal{M}ax(\mathcal{S}_2) = \{ \quad \}$$

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$$\mathcal{M}ax(\mathcal{S}_1) = \{a_0 a_1 a_2 a_3\}$$

$$\mathcal{M}ax(\mathcal{S}_2) = \{a_0 \qquad\qquad\qquad\}$$

$$\mathcal{M}ax(\mathcal{S}_3) = \{ \qquad\qquad\qquad\qquad \}$$

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Let's play on $[\![((a \supset a) \supset a) \supset a]\!]$

It is $\bullet$'s turn

$$a_3 \longrightarrow a_2 \longrightarrow a_1 \overset{\bullet}{\longrightarrow} a_0$$

$$\mathcal{M}ax(\mathcal{S}_1) = \{a_0 a_1 a_2 a_3\}$$

$$\mathcal{M}ax(\mathcal{S}_2) = \{a_0 a_1 \qquad\qquad\}$$

$$\mathcal{M}ax(\mathcal{S}_3) = \{ \qquad\qquad\qquad \}$$

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Let's play on $[\![((a \supset a) \supset a) \supset a]\!]$

It is $\circ$'s turn

$$a_3 \longrightarrow a_2 \longrightarrow a_1 \longrightarrow a_0^{\circ}$$

$$\mathcal{M}ax(\mathcal{S}_1) = \{a_0 a_1 a_2 a_3\}$$

$$\mathcal{M}ax(\mathcal{S}_2) = \{a_0 a_1 a_2 a_1 a_2 a_3\}$$

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$$a_3 \longrightarrow a_2 \longrightarrow a_1 \overset{\bullet}{\longrightarrow} a_0$$

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(dotted lines identify the justifier of a move)

Theorem (Full Completeness)

Every WIS *on* $[\![F]\!]$ *is the image of a proof of* F.

We can translate derivations into WISs:

$$\left\{\!\!\left\{\frac{}{a^\bullet \vdash a^\circ}\,\mathsf{AX}\right\}\!\!\right\} = \{\epsilon, a^\circ, a^\circ a^\bullet\}$$

$$\left\{\!\!\left\{\frac{\begin{array}{c}\|\mathfrak{D}_1\\ \Gamma^\bullet \vdash B^\circ\end{array}}{\Gamma^\bullet, A^\bullet \vdash B^\circ}\,\mathsf{W}\right\}\!\!\right\} = \left\{\!\!\left\{\frac{\begin{array}{c}\|\mathfrak{D}_1\\ \Gamma^\bullet A^\bullet \vdash B^\circ\end{array}}{\Gamma^\bullet, \vdash A^\bullet \supset B^\circ}\,\supset^{\mathsf{R}}\right\}\!\!\right\} = \left\{\!\!\left\{\frac{\begin{array}{c}\|\mathfrak{D}_1\\ \Gamma^\bullet, A^\bullet, C^\bullet \vdash B^\circ\end{array}}{\Gamma^\bullet, A^\bullet \wedge C^\bullet \vdash B^\circ}\,\wedge^{\mathsf{L}}\right\}\!\!\right\} = \{\!\{\mathfrak{D}_1\}\!\}$$

$$\left\{\!\!\left\{\frac{\begin{array}{c}\|\mathfrak{D}_1\\ \Gamma^\bullet \vdash A^\circ\end{array} \quad \begin{array}{c}\|\mathfrak{D}_2\\ \Delta^\bullet \vdash B^\circ\end{array}}{\Gamma^\bullet, \Delta^\bullet \vdash (A \wedge B)^\circ}\,\wedge^{\mathsf{R}}\right\}\!\!\right\} \qquad \{\!\{\mathfrak{D}_1\}\!\} \cup \{\!\{\mathfrak{D}_2\}\!\} = \left\{\!\!\left\{\frac{\begin{array}{c}\|\mathfrak{D}_1\\ \Gamma^\bullet, A_1^\bullet, A_2^\bullet \vdash B^\circ\end{array}}{\Gamma^\bullet, A^\bullet \vdash B^\circ}\,\mathsf{C}\right\}\!\!\right\} = \text{identify moves } \{\!\{\mathfrak{D}_1\}\!\} \text{ whenever they are in } A_1^\bullet = A_2^\bullet$$

$$\left\{\!\!\left\{\frac{\begin{array}{c}\|\mathfrak{D}_1\\ \Gamma^\bullet \vdash A^\circ\end{array} \quad \begin{array}{c}\|\mathfrak{D}_2\\ \Delta^\bullet, B^\bullet \vdash C^\circ\end{array}}{\Gamma^\bullet, \Delta^\bullet, (A \supset B)^\bullet \vdash C^\circ}\,\supset^{\mathsf{L}}\right\}\!\!\right\} = \mathfrak{D}_2 \cup \text{“if } \circ \text{ starts to play on } A\text{, then continue as in } \{\!\{\mathfrak{D}_1\}\!\}\text{”}$$

Theorem (Full Completeness)

Every WIS on $[\![F]\!]$ is the image of a proof of F.

Sequent	Shape of $\mathcal{S}$	Shape of $\mathfrak{D}_{\mathcal{S}}$
$\vdash 1$	$\mathcal{S} = \{\epsilon\}$	$\dfrac{}{\vdash 1}\ 1$
$a \vdash a$	$\mathcal{S} = \{\epsilon, a, aa\}$	$\dfrac{}{a \vdash a}\ \mathrm{AX}$
$\Gamma, B \wedge C \vdash A$	any	$\dfrac{\overset{\mathfrak{D}_{\mathcal{S}} \Vert}{\Gamma, B, C \vdash A}}{\Gamma, B \wedge C \vdash A}\ \wedge^{\mathrm{L}}$
$\Gamma \vdash B \supset A$	any	$\dfrac{\overset{\mathfrak{D}_{\mathcal{S}} \Vert}{\Gamma, B \vdash A}}{\Gamma \vdash B \supset A}\ \supset^{\mathrm{R}}$
$\Gamma \vdash A_1 \wedge A_2$ Γ contains no $\wedge$-formula	$\mathcal{S} = \mathcal{T} \cup \mathcal{R}$ $\mathcal{T} = \{\tau \in \mathcal{S} \mid \tau$ contains no moves in $A_2\}$ $\mathcal{R} = \{\rho \in \mathcal{S} \mid \rho$ contains no moves in $A_1\}$	$\dfrac{\dfrac{\overset{\mathfrak{D}_{\mathcal{T}} \Vert}{\Gamma \vdash A_1} \quad \overset{\mathfrak{D}_{\mathcal{R}} \Vert}{\Gamma \vdash A_2}}{\Gamma, \Gamma \vdash A_1 \wedge A_2}\ \wedge^{\mathrm{R}}}{\Gamma \vdash A_1 \wedge A_2}\ \mathrm{C}$
$\Gamma, A \supset B\{c^\bullet\} \vdash c^\circ$ c atomic and $A \supset B\{c^\bullet\} \neq c^\bullet$ $B\{c^\bullet\}$ contains the atom $c^\bullet$ Γ contains no $\wedge$-formulas	$c^\circ c^\bullet \in \mathcal{S}$ $\mathcal{T} = \{\tau \mid$ there are σ and τ' such that $\sigma\tau\tau' \in \mathrm{Split}^{A}_{\mathcal{S}}\}$ $\mathcal{R} = \{\rho \mid$ there is no σ such that $\rho\sigma \in \mathrm{Split}^{A}_{\mathcal{S}}\}$	$\dfrac{\dfrac{\overset{\mathfrak{D}_{\mathcal{T}} \Vert}{\Gamma \vdash A} \quad \overset{\mathfrak{D}_{\mathcal{R}} \Vert}{\Gamma, A \supset B\{c^\bullet\}, B\{c^\bullet\} \vdash c^\circ}}{\Gamma, \Gamma, A \supset B\{c^\bullet\}, A \supset B\{c^\bullet\} \vdash c^\circ}\ \supset^{\mathrm{L}}}{\Gamma, A \supset B\{c^\bullet\} \vdash c^\circ}\ \mathrm{C}$
$\Gamma, B \vdash A$	$\mathcal{S}$ contains no moves in B	$\dfrac{\overset{\mathfrak{D}_{\mathcal{S}} \Vert}{\Gamma \vdash A}}{\Gamma, B \vdash A}\ \mathrm{W}$

Proof equivalence in LI

$$
\mathfrak{D} = \dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{ax}}{\vdash b \supset b}\supset^{\mathsf{R}} \qquad \dfrac{\dfrac{\dfrac{}{a \vdash a_0}\,\mathsf{ax} \quad \dfrac{}{a \vdash a_2}\,\mathsf{ax}}{a, a \vdash a_0 \wedge a_2}\wedge}{a \vdash a_0 \wedge a_2}\,\mathsf{C}}{(b \supset b) \supset a \vdash a_0 \wedge a_2}\supset^{\mathsf{L}}}{\vdash ((b \supset b) \supset a) \supset (a_0 \wedge a_2)}\supset^{\mathsf{R}}
\quad \underset{\text{non-local}}{\sim} \quad
\mathfrak{D}' = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{ax}}{\vdash b \supset b}\supset^{\mathsf{R}} \quad \dfrac{}{a \vdash a_0}\,\mathsf{ax}}{(b \supset b) \supset a \vdash a_0}\supset^{\mathsf{L}} \quad \dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{ax}}{\vdash b \supset b}\supset^{\mathsf{R}} \quad \dfrac{}{a \vdash a_2}\,\mathsf{ax}}{(b \supset b) \supset a \vdash a_2}\supset^{\mathsf{L}}}{(b \supset b) \supset a, (b \supset b) \supset a \vdash a_0 \wedge a_2}\wedge}{(b \supset b) \supset a \vdash a_0 \wedge a_2}\,\mathsf{C}}{\vdash ((b \supset b) \supset a) \supset (a_0 \wedge a_2)}\supset^{\mathsf{R}}
$$

and

$$
\{\!\{\mathfrak{D}\}\!\} = \{\!\{\mathfrak{D}'\}\!\} = \left\{ \epsilon, \begin{array}{c} a_0 \ , \ a_0 a \ , \ a_0 ab \ , \ a_0 abb \\ \\ a_2 \ , \ a_2 a \ , \ a_2 ab \ , \ a_2 abb \end{array} \right\}
$$

which corresponds to the lambda term $\lambda f^{(b \supset b) \supset a}.(f(\lambda x^a.x), f(\lambda y^a.y))$

Independent rules

$$\dfrac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$$

$$\dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1 \qquad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$$

Resource Management

$$\dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A,B\vdash C}2\times \mathrm{C}}{\Gamma,A\wedge B\vdash C}\wedge^{\mathrm{L}} \equiv_{\mathrm{c}} \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}2\times\wedge^{\mathrm{L}}}{\Gamma,A\wedge B\vdash C}\mathrm{C} \qquad \dfrac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}2\times \mathrm{W}}{\Gamma,A\wedge B\vdash C}\wedge^{\mathrm{L}} \equiv_{\mathrm{c}} \dfrac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}\mathrm{W}$$

$$\dfrac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}\mathrm{C}}{\Gamma,A,A\vdash B}\mathrm{W} \equiv_{\mathrm{c}} \Gamma,A,A\vdash B \qquad \dfrac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}\mathrm{W}}{\Gamma,A\vdash B}\mathrm{C} \equiv_{\mathrm{c}} \Gamma,A\vdash B$$

Excising and Unfolding

$$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta\vdash C}{B,\Delta\vdash C}\mathrm{W}}{\Gamma,\Delta,A\supset B\vdash C}\supset^{\mathrm{L}} \equiv_{\mathrm{e}} \dfrac{\Delta\vdash C}{\Gamma,\Delta,A\supset B\vdash C}\mathrm{W} \quad \Bigg\| \quad \dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}\mathrm{C}}{\Gamma,A\supset B\vdash C}\supset^{\mathrm{L}} \equiv_{\mathrm{u}} \dfrac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\supset^{\mathrm{L}}}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\supset^{\mathrm{L}}}{\Gamma,\Delta,A\supset B\vdash C}\mathrm{C}$$

$$\equiv_{\mathrm{WIS}} := (\equiv \cup \equiv_{\mathrm{c}} \cup \equiv_{\mathrm{e}} \cup \equiv_{\mathrm{u}})$$

The permutation $\equiv_{\mathrm{u}}$ is said *non-local*

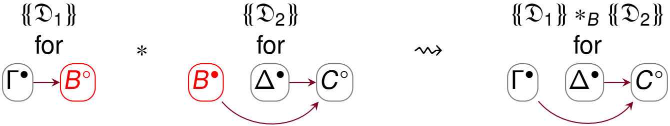
What about cut?

Composition = Interaction + Hide

$$\dfrac{\dfrac{\mathfrak{D}_1}{\Gamma \vdash B} \quad \dfrac{\mathfrak{D}_2}{\Delta, B \vdash C}}{\Gamma, \Delta \vdash C}\ \text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\mathfrak{D}_1}{\Gamma \vdash B} \quad \dfrac{\mathfrak{D}_2}{\Delta, B \vdash C}}{\Gamma, \Delta, B \supset B \vdash C}\ \supset^{L}}{\Gamma, \Delta \vdash C}\ \text{hide}$$

Composition = Interaction + Hide

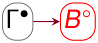
$$\dfrac{\dfrac{\mathfrak{D}_1}{\Gamma \vdash B} \qquad \dfrac{\mathfrak{D}_2}{\Delta, B \vdash C}}{\Gamma, \Delta \vdash C}\ \text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\mathfrak{D}_1}{\Gamma \vdash B} \qquad \dfrac{\mathfrak{D}_2}{\Delta, B \vdash C}}{\Gamma, \Delta, B \supset B \vdash C}\ \supset^{L}}{\Gamma, \Delta \vdash C}\ \text{hide}$$



Composition = Interaction + Hide

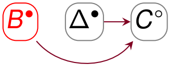
$$\dfrac{\dfrac{\mathfrak{D}_1}{\Gamma \vdash B} \qquad \dfrac{\mathfrak{D}_2}{\Delta, B \vdash C}}{\Gamma, \Delta \vdash C}\ \text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\mathfrak{D}_1}{\Gamma \vdash B} \qquad \dfrac{\mathfrak{D}_2}{\Delta, B \vdash C}}{\Gamma, \Delta, B \supset B \vdash C}\ \supset^L}{\Gamma, \Delta \vdash C}\ \text{hide}$$

$\{\!\{\mathfrak{D}_1\}\!\}$ for



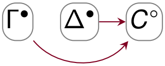
$*$

$\{\!\{\mathfrak{D}_2\}\!\}$ for

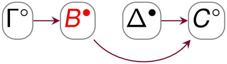


$\rightsquigarrow$

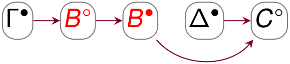
$\{\!\{\mathfrak{D}_1\}\!\} *_B \{\!\{\mathfrak{D}_2\}\!\}$ for



Literature approach



Approach we use here



Interaction

Let τ a view over $A \vdash B_1$ and ρ a view over $B_2 \vdash C$.

The *interaction* of τ ad ρ over B is the sequence of moves $\sigma = \tau \overset{B}{\mathbin{\circ\!\!\!\bullet}} \rho$ starting with $\sigma_0 = \rho_0$ defined as:

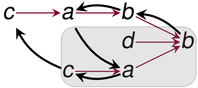
$$\sigma_{i+1} = \begin{cases} \tau_{k+1} & \text{where } \sigma_i = \tau_k \text{ is a move in } A \text{ or a } \circ\text{-move in } B_1 \\ \rho_{k+1} & \text{where } \sigma_i = \rho_k \text{ is a move in } C \text{ or a } \circ\text{-move in } B_2 \\ b^\perp & \text{where } \sigma_i = b \text{ is a } \bullet\text{-move in } B_1 \text{ and } b^\perp \text{ occurs in } \rho \\ b^\perp & \text{where } \sigma_i = b \text{ is a } \bullet\text{-move in } B_2 \text{ and } b^\perp \text{ occurs in } \tau \\ \text{not defined} & \text{otherwise} \end{cases}$$

Interaction

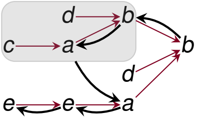
$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

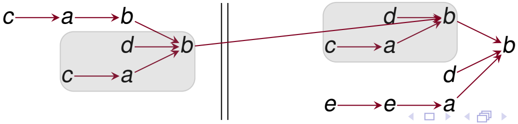


$\tau = bbaacc$



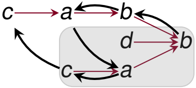
$\rho_1 = bbaaee$

$\tau \overset{B}{\circ} \rho_1 = b$

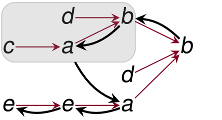


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!] \;\|\; [\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

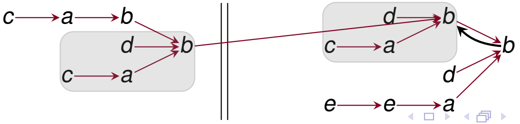


$\tau = bbaacc$



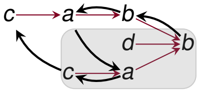
$\rho_1 = bbaaee$

$$\tau \overset{B}{\circledcirc} \rho_1 = bb$$

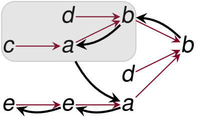


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

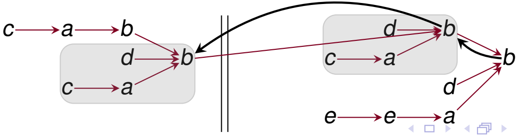


$\tau = bbaacc$



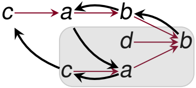
$\rho_1 = bbaaee$

$\tau \overset{B}{\circledcirc} \rho_1 = bbb$

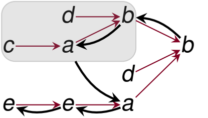


Interaction

$[\![((c \supset a) \supset b) \vdash \color{red}{d \supset (c \supset a) \supset b}]\!]$ || $[\![\color{red}{d \supset (c \supset a) \supset b}, d \vdash ((e \supset e) \supset a) \supset b]\!]$

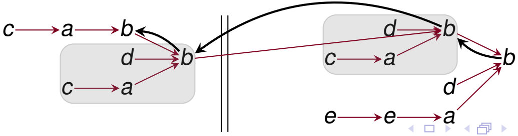


$\tau = bbaacc$



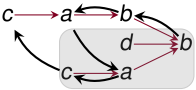
$\rho_1 = bbaaee$

$\tau \overset{B}{◐} \rho_1 = bbbb$

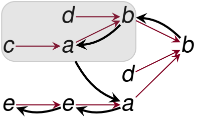


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

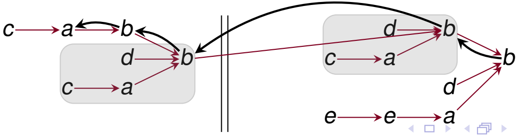


$\tau = bbaacc$



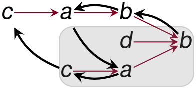
$\rho_1 = bbaaee$

$\tau \overset{B}{\mathbin{◐}} \rho_1 = bbbba$

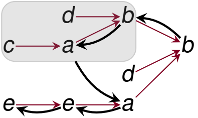


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

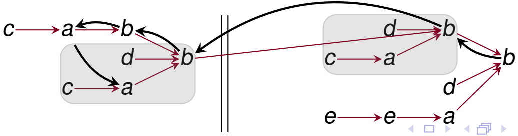


$\tau = bbaacc$



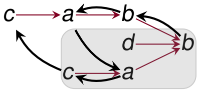
$\rho_1 = bbaaee$

$\tau \overset{B}{\circ} \rho_1 = bbbbaa$

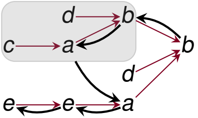


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

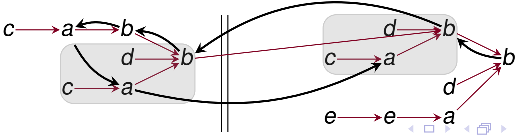


$\tau = bbaacc$



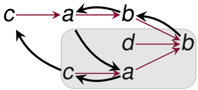
$\rho_1 = bbaaee$

$\tau \overset{B}{\circledcirc} \rho_1 = bbbbaaa$

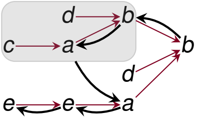


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

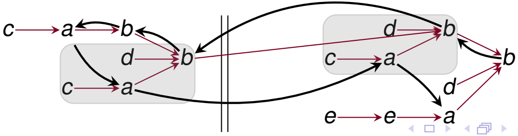


$\tau = bbaacc$



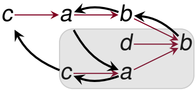
$\rho_1 = bbaaee$

$\tau \overset{B}{\bullet} \rho_1 = bbbbaaaa$

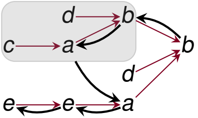


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

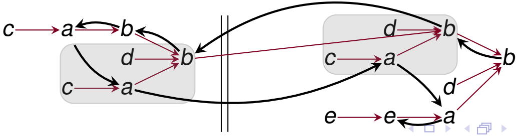


$\tau = bbaacc$



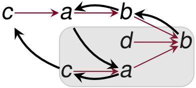
$\rho_1 = bbaaee$

$\tau \overset{B}{◐} \rho_1 = bbbbaaaae$

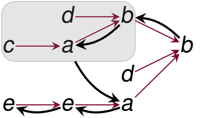


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

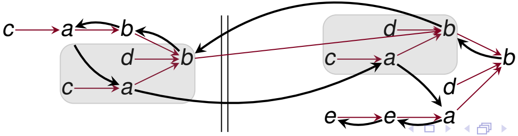


$\tau = bbaacc$



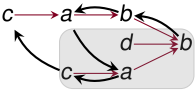
$\rho_1 = bbaaee$

$\tau \overset{B}{\circledcirc} \rho_1 = bbbbaaaaee$

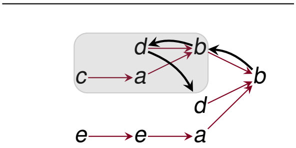


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!] \quad || \quad [\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

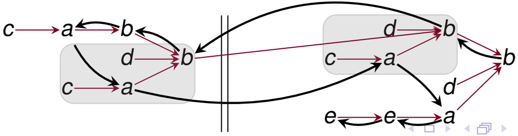


$\tau = bbaacc$



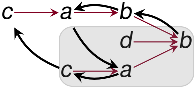
$\rho_2 = bbdd$

$\tau \overset{B}{\LEFTcircle} \rho_2 = b$

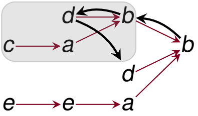


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

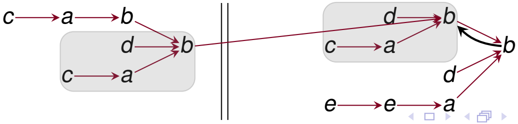


$\tau = bbaacc$



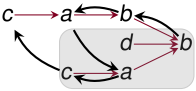
$\rho_2 = bbdd$

$\tau \overset{B}{\circ} \rho_2 = bb$

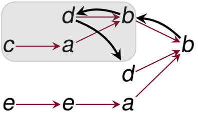


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

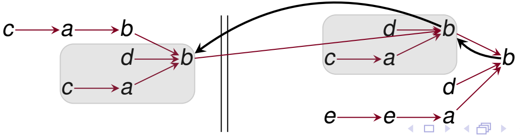


$\tau = bbaacc$



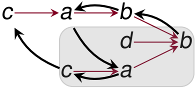
$\rho_2 = bbdd$

$\tau \overset{B}{\mathbin{◐}} \rho_2 = bbb$

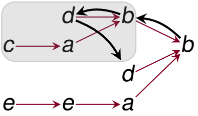


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!] \quad \Big\| \quad [\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

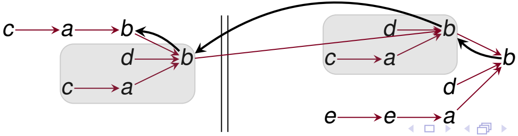


$\tau = bbaacc$



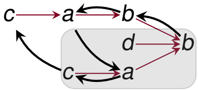
$\rho_2 = bbdd$

$\tau \overset{B}{◐} \rho_2 = bbbb$

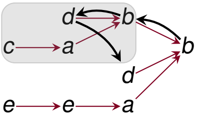


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!] \;\Big\|\; [\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

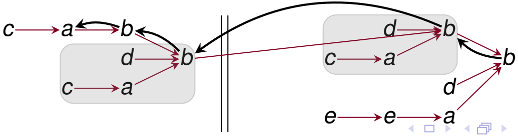


$\tau = bbaacc$



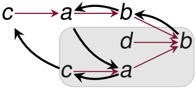
$\rho_2 = bbdd$

$\tau \overset{B}{\circ} \rho_2 = bbbba$

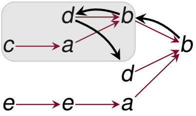


Interaction

$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$

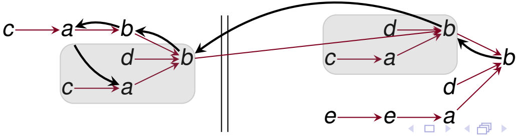


$\tau = bbaacc$



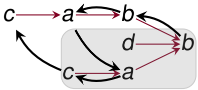
$\rho_2 = bbdd$

$\tau \overset{B}{\bullet} \rho_2 = bbbbaa$

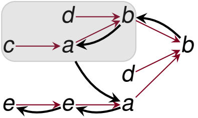


Interaction

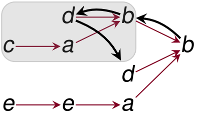
$[\![((c \supset a) \supset b) \vdash d \supset (c \supset a) \supset b]\!]$ || $[\![d \supset (c \supset a) \supset b, d \vdash ((e \supset e) \supset a) \supset b]\!]$



$\tau = bbaacc$

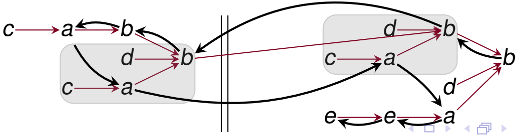


$\rho_1 = bbaaee$



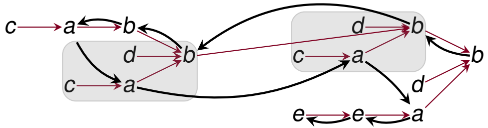
$\rho_2 = bbdd$

$\tau \overset{B}{\bullet} \rho_2 = bbbbaa$ is a prefix of $\tau \overset{B}{\bullet} \rho_1 = bbbbaaaaee$



Hide

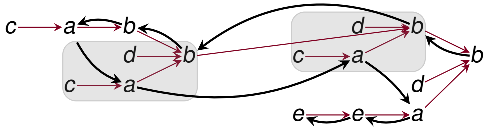
$$\tau \overset{B}{\LEFTcircle} \rho_1 = bbbbaaaaee$$



Justifiers are defined as in τ and in ρ. If a move is justied by an hidden move, than follow the justification arrow (dotted lines) until reach an unhidden move

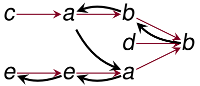
Hide

$$\tau \overset{B}{\LEFTcircle} \rho_1 = bbbbaaaaee$$



$$\downarrow$$

$$\tau \overset{B}{*} \rho_1 = bbaaee$$



Justifiers are defined as in τ and in ρ. If a move is justied by an hidden move, than follow the justification arrow (dotted lines) until reach an unhidden move

Theorem (Compositionality)

The composition of two WISs *is a* WIS.

Theorem

There is a one-to-one correspondence between the following sets:

- *the set of* WISs *over* $[\![F]\!]$
- *the set of normal derivations of F in Natural deduction*
- *the set of* $\eta\beta$*-normal (simply typed)* λ*-terms (with pairs) of type F*
- *the set of* (cut-*free) derivations of F in* LI *modulo* $\equiv_{\mathrm{WIS}}$
- *the set of morphisms of a (small) Cartesian Closed Category*