Course Notes
"An Introduction to Proof Equivalence"

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1 The Normalization Criterion in Natural Deduction

In the previous classes we introduced two criteria for proof equivalence, the normalization and permutation criteria. Natural questions concerning them are:

1. Are the two criteria equivalent?
2. Is there a mathematical interpretation of proofs such that equivalent derivations have the same interpretation?
3. Are these criteria decidable? In polynomial time?

Today we focus on the normalization criterion. As we said, this criterion is best suited for natural deduction, and this in turn is best suited for intuitionistic, rather than classical, logic. So, today we focus on natural deduction for intuitionistic logic, that is, on the proof system **NJ** presented on Monday.

1.1 The Paradise of the "Negative" Fragment

In this subsection we present natural deduction *at its best*. Yesterday, Matteo introduced you to the "paradise" of multiplicative linear logic with the elegant approach based on proof nets. Yet, he also showed how easy it is to fall from Heaven. Similarly, today we will see how the "paradise" of a certain fragment of **NJ** motivates the approach on which the normalization criterion is based, but this approach falls short as soon as we look outside of this fragment.

What I call the "negative" fragment of **NJ** is obtained by simply restricting the language to just two connectives, namely $\wedge$ and $\to$. The *negative language* $\mathcal{L}_{\to,\wedge}$ is thus formed by the formulas belonging to the following grammar:

$$A ::= p \mid A \to A \mid A \wedge A$$

Incidentally, this language is really the core of the Curry-Howard correspondence with λ-calculus that Giulio Guerrieri is discussing in his course. Just a matter of chance? Not really! Since this is actually the part of logic for which natural deduction works decently well, as we will see.

The natural deduction proof system for $\mathcal{L}_{\to,\wedge}$ is just the restriction of **NJ** containing only the intro- and elim-rules for the connectives $\to$, $\wedge$, as well as the two associated reduction rules, that we recall below:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\A\end{matrix}\quad\begin{matrix}\Pi_2\\B\end{matrix}}{A\wedge B}\wedge\mathrm{I}}{A}\wedge\mathrm{E}_1 \quad\rightsquigarrow\quad \begin{matrix}\Pi_1\\A\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\overset{i}{A}\\\Pi_1\\B\end{matrix}}{A\to B}\to\mathrm{I}(i)\qquad\begin{matrix}\Pi_2\\A\end{matrix}}{B}\to\mathrm{E} \quad\rightsquigarrow\quad \begin{matrix}\Pi_2\\[A]\\\Pi_1\\B\end{matrix}$$

Furthermore, one usually considers also further *expansion rules*, namely:

$$A \overset{i}{\wedge} B \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{i}{A\wedge B}}{A}\wedge\mathrm{E}_1\qquad\dfrac{\overset{i}{A\wedge B}}{B}\wedge\mathrm{E}_1}{A\wedge B}\wedge\mathrm{I}$$

$$A \overset{i}{\to} B \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{i}{A\wedge B}\qquad\overset{i}{A}}{B}\to\mathrm{E}}{A\to B}\to\mathrm{I}(i)$$

Reductions and expansions can be seen as somehow dual operations. Usually, the reduction rules are explained via the so-called *inversion principle*:

> The consequences that one can draw from a logically complex propositions are *included* in the conditions needed to conclude that such a proposition is true.

The expansion rules can then be explained via the dual *stability principle*:

> The consequences that one can draw from a logically complex propositions *include* all the conditions needed to conclude that such a proposition is true.

Together, these principles say that introduction and elimination rules, so to say, perfectly match each other.

Hence, $\mathbf{NJ}_{\to,\wedge}$ includes four construction rules, plus two reduction and two expansion rules, that's it.

We already mentioned the fundamental result of **NJ**, that we recall below:

Theorem 1. *For any derivation* Π*, there exists a* unique *normal derivation* $\mathrm{nf}(\Pi)$ *such that* $\Pi \rightsquigarrow^* \mathrm{nf}(\Pi)$.

There is a reason why *normal* derivations are very important in natural deduction. This is the fact that these formulas satisfy the so-called *sub-formula property*:

Proposition 2. *In a normal derivation* Π *of* $\mathbf{NJ}_{\to,\wedge}$*, any formula occurring in* Π *is either a sub-formula of one of the hypotheses or a sub-formula of the conclusion.*

In other words, in a normal derivation we are only working with the "tools" that we were given, i.e. the hypotheses and the conclusion to prove. Beware that this is *not* how people prove things, usually: think of people in number theory, who, in order to prove facts about, say, prime numbers, rely on super-difficult results from complex analysis! This is indeed the magic of normalization or cut-elimination theorems, indeed: showing that arbitrary proofs, using concepts from many different areas, can, at least *in principle*, be transformed into proofs only using the concepts involved in the statement that are actually proved.

Notice that in Proposition 2 I mentioned derivations in $\mathbf{NJ}_{\to,\wedge}$ and not in full **NJ**. This was not for chance: as we will see to have such a result for full **NJ** we will have to add *new* reductions. Indeed, if a normal derivation does not satisfy the sub-formula, this means that it is not *truly* normal: it rather means that some reduction rule is missing!

As far as we stick to $\mathbf{NJ}_{\to,\wedge}$ we thus have unicity of normal forms, and these satisfy sub-formulas. At this point, we might ask ourselves, how canonical is this apparently canonical system?

There are a few important results that suggest that $\mathbf{NJ}_{\to,\wedge}$ is indeed close to the best that we can expect. First, we have results from *denotational semantics*, something that Matteo will discuss later. But let's anticipate on this. There are two kinds of standard semantics for **NJ**:

Set-theoretic semantics : here each formula A is translated into a set $[\![A]\!]$, so that $[\![A \wedge B]\!] = [\![A]\!] \times [\![B]\!]$ and $[\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}$; each proof $\Pi : A_1, \dots, A_n \vdash B$ is turned then into a set-theoretic function $[\![\Pi]\!] : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \longrightarrow [\![B]\!]$. Then we have the following nice result, due to Statman .

> **Theorem 3** (cf. [2, 8])**.** *For any interpretation* $[\![-]\!]$*, for which propositional variables are interpreted as infinite sets, for any two derivations* Π, Π' *of* $\mathbf{NJ}_{\to,\wedge}$*,* $[\![\Pi]\!] = [\![\Pi']\!]$ *iff* $\mathrm{nf}(\Pi) = \mathrm{nf}(\Pi')$.

Category-theoretic semantics : this is just for those of you who know something about category theory. Think of a category as an algebraic axiomatization of the notion of sets and functions. Now, a category in which it makes sense to talk of the cartesian product of two objects, and of the function space between two objects is called a *cartesian closed category*. In any such category $\mathcal{C}$ we can interpret each formula A as an object $[\![A]\!]_{\mathcal{C}}$ in such a way that $[\![A \wedge B]\!]_{\mathcal{C}} = [\![A]\!]_{\mathcal{C}} \times [\![B]\!]_{\mathcal{C}}$ and $[\![A \to B]\!]_{\mathcal{C}} =_{\mathcal{C}} ([\![A]\!]_{\mathcal{C}}, [\![B]\!]_{\mathcal{C}})$; each proof $\Pi : A_1, \dots, A_n \vdash B$ yields then a *morphism* in this category $[\![\Pi]\!]_{\mathcal{C}} : [\![A_1]\!]_{\mathcal{C}} \times \cdots \times [\![A_n]\!]_{\mathcal{C}} \longrightarrow [\![B]\!]_{\mathcal{C}}$. Then we have the following very strong result.

> **Theorem 4** (cf. [3])**.** *For any two derivations* Π, Π' *of* $\mathbf{NJ}_{\to,\wedge}$*,*
>
> $$\Big(\textit{For all cartesian closed category } \mathcal{C},\ [\![\Pi]\!]_{\mathcal{C}} = [\![\Pi']\!]_{\mathcal{C}}\Big) \quad \textit{iff} \quad \mathrm{nf}(\Pi) = \mathrm{nf}(\Pi').$$

Beyond denotational methods, there is a way to establish the canonicity of $\mathbf{NJ}_{\to,\wedge}$ which does not rely on external interpretations, that is, purely syntactic. To do this, we first need to define a class of well-behaved equivalences on derivations. These are the equivalences which are stable by substitution, as defined below:

Definition 1. *Let* $\mathcal{E} = ({}^{\Gamma\vdash A}\!\equiv)_{\Gamma\vdash A}$ *be a family of equivalence relations, where* ${}^{\Gamma\vdash A}\!\equiv$ *is an equivalence over derivations of hypotheses* Γ *and conclusion* A*.* $\mathcal{E}$ *is called a* stable equivalence *if it satisfies the following conditions: for all derivations* $\Pi, \Pi' : \Gamma \vdash A$*, formula* B *and propositional variable* p*,*

$$\Pi \ {}^{\Gamma\vdash A}\!\equiv \Pi' \quad \Rightarrow \quad \Pi[B/p] \ {}^{(\Gamma\vdash A)[B/p]}\!\equiv \Pi'[B/p]. \tag{*}$$

The *trivial* equivalence is the one for which *any* two derivations with same hypotheses and conclusion are related. This is obviously stable. More interestingly, the equivalences arising from either the normalization of the permutation criterion are also stable.

An interesting question is whether there exists some *intermediate* stable equivalence in between normalization-equivalence and the trivial equivalence. This would mean that we can consistently add *new* proof equivalences to normalization-equivalence. The following result shows that this is indeed not possible.

Theorem 5 (Separation or Böhm's theorem, cf. [7, 1])**.** *Normalization equivalence is a* maximum non-trivial stable equivalence *on* $\mathbf{NJ}_{\to,\wedge}$*-derivations. More precisely, for any non-trivial stable equivalence* $\mathcal{E} = ({}^{\Gamma\vdash A}\!\equiv)_{\Gamma\vdash A}$ *extending normalization-equivalence and any two derivations* $\Pi, \Pi' : \Gamma \vdash A$*,*

$$\Pi \;{}^{\Gamma\vdash A}\!\equiv\; \Pi' \quad \Rightarrow \quad \mathrm{nf}(\Pi) = \mathrm{nf}(\Pi').$$

Proof sketch. The core of the proof is showing that if $\Pi \;{}^{\Gamma\vdash A}\!\equiv\; \Pi'$ and $\mathrm{nf}(\Pi) \neq \mathrm{nf}(\Pi')$, then for any two other derivations $\Sigma, \Sigma' : \Delta \vdash B$, one can find a substitution $\theta = [B_1/p_1, \dots, B_k/p_k]$ and derivations $\Theta_i : \Delta \vdash C_i\theta$ (for any C_i in Γ) and $\Theta : A\theta \vdash B$ such that:

$$\mathrm{nf}\begin{pmatrix}\Delta\\ \dots\Theta_i\dots\\ [\Gamma\theta]\\ \Pi\theta\\ [A\theta]\\ \Theta\\ B\end{pmatrix} = \mathrm{nf}\begin{pmatrix}\Delta\\ \Sigma\\ B\end{pmatrix} \qquad \text{and} \qquad \mathrm{nf}\begin{pmatrix}\Delta\\ \dots\Theta_i\dots\\ [\Gamma\theta]\\ \Pi'\theta\\ [A\theta]\\ \Theta\\ B\end{pmatrix} = \mathrm{nf}\begin{pmatrix}\Delta\\ \Sigma'\\ B\end{pmatrix}.$$

This means that, if ever $\mathcal{E}$ strictly extends normalization-equivalence, then we can identify, through $\mathcal{E}$, any two derivations, and thus $\mathcal{E}$ is the trivial equivalence. Actually, to prove this result it is enough to take as Σ, Σ' the two derivations

$$\Pi_1 := \dfrac{\dfrac{\overset{0}{p}}{p\to p}\to\mathrm{I}(1)}{p\to(p\to p)}\to\mathrm{I}(0) \qquad \Pi_2 := \dfrac{\dfrac{\overset{0}{p}}{p\to p}\to\mathrm{I}(0)}{p\to(p\to p)}\to\mathrm{I}(1).$$

since, for any two other $\Sigma, \Sigma' : \Delta \vdash B$, one can construct a derivation $\Sigma^* : B \to (B \to B), \Delta \vdash B$ such that

$$\begin{matrix}\Pi_1[B/p]\\ [B\to(B\to B)]\\ \Sigma^*\\ B\end{matrix}\quad \Delta \quad \rightsquigarrow \quad \begin{matrix}\Delta\\ \Sigma\\ B\end{matrix} \qquad \text{and} \qquad \begin{matrix}\Pi_2[B/p]\\ [B\to(B\to B)]\\ \Sigma^*\\ B\end{matrix}\quad \Delta \quad \rightsquigarrow \quad \begin{matrix}\Delta\\ \Sigma'\\ B\end{matrix}$$

Indeed, we can let

$$\Sigma^* := \dfrac{\dfrac{B\to(\overset{i}{B}\to B) \qquad \begin{matrix}\Delta\\ \Sigma\\ B\end{matrix}}{B\to B}\to\mathrm{E} \qquad \begin{matrix}\Delta\\ \Sigma'\\ B\end{matrix}}{B}\to\mathrm{E}.$$

□

1.2 Paradise Lost: Full Intuitionistic Logic

Yesterday we have seen that eating a small fruit can make you fall from Heaven...it is enough to add seemingly harmless operations like the multiplicative units or the additive connectives to undermine the possibility of having a canonical proof system. We are now going to see that something very similar happens when we extend the negative fragment with other connectives.

Let us recall the rules for disjunction and the false:

$$\dfrac{A_i}{A_0\vee A_1}\vee\mathrm{I}_i \qquad \dfrac{A\vee B \qquad \begin{matrix}\overset{i}{A}\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}\overset{j}{B}\\ \vdots\\ C\end{matrix}}{C}\vee\mathrm{E}(i,j) \qquad \dfrac{\mathbf{F}}{A}\,\mathbf{F}\mathrm{E}$$

Observe that additive conjunction and disjunction in linear logic are very close to conjunction and disjunction in intuitionistic logic:

$$\dfrac{\Gamma\vdash A_i}{\Gamma\vdash A_0\oplus A_1}\oplus\mathrm{R}_i \qquad \dfrac{\Gamma\vdash A \qquad \Gamma\vdash B}{\Gamma\vdash A\&B}\,\&\mathrm{R}$$

where the &R-rule can be rewritten, dually, as a $\oplus$L-rule:

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}\ \oplus\mathrm{E}$$

From what we have seen yesterday, one may expect that by adding disjunction to $\mathbf{NJ}_{\to,\wedge}$ something may go wrong.

Moreover, Matteo yesterday talked about multiplicative units, and he did not even dare to talk about *additive* units $\top$ and $\mathbf{0}$, which are even worse to handle. Yet, the additive false $\mathbf{0}$ is very close to the intuitionistic false, since it is regulated by the rule:

$$\frac{}{\Gamma, \mathbf{0} \vdash}\ \mathbf{0}$$

which is essentially the same as the rule for $\mathbf{F}$ in $\mathbf{LJ}$.

Let us recall reductions and expansions for disjunction:

$$\frac{\dfrac{A_i}{A_0 \vee A_1}\vee\mathrm{I}_i \qquad \overset{i}{A} \quad \overset{j}{B} \\ \vdots \quad \vdots \\ C \quad C}{C}\vee\mathrm{E}(i,j) \qquad \rightsquigarrow \qquad \begin{array}{c}[A_i]\\ \vdots \\ C\end{array}$$

$$A \overset{k}{\vee} B \qquad \rightsquigarrow \qquad \frac{A \overset{k}{\vee} B \qquad \dfrac{\overset{i}{A}}{A \vee B}\vee\mathrm{I}_1 \qquad \dfrac{\overset{j}{B}}{A \vee B}\vee\mathrm{I}_2}{A \vee B}\vee\mathrm{E}(i,j)$$

One can actually define a more general notion of expansion, which however is much more problematic and we will not discuss it here.

Permutative Conversions But first, let us start from the good news. As said before, a normal derivation should be one which does satisfy the sub-formula property. However, in presence of disjunction, one can construct derivations without redundancies which do not satisfy this property:

$$\frac{\overset{0}{A \vee A} \qquad \dfrac{\overset{1}{A}}{C \to A}\to\mathrm{I}(3) \qquad \dfrac{\overset{2}{A}}{C \to A}\to\mathrm{I}(3)}{C \to A}\vee\mathrm{E}(1,2) \qquad \overset{4}{C}}{A}\to\mathrm{E}$$

This derivation has hypotheses $A \vee A$ and C, and conclusion A, and has no local peak. Yet, it contains the formula $C \to A$ which is *not* a subformula of either an hypothesis or the conclusion. This derivation is *not* normal, some *new* reduction rule is missing. The standard solution is to add so-called *permutative conversions*, i.e. new reductions of the form

$$\frac{\begin{array}{ccc}\overset{i}{A} & \overset{j}{B} \\ \Pi_1 & \Pi_2 \\ A \vee B \qquad C & C\end{array}}{\dfrac{C}{D}\,\mathrm{R}}\vee\mathrm{E}(i,j) \qquad \rightsquigarrow \qquad \frac{\begin{array}{ccc}\overset{i}{A} & \overset{j}{B} \\ \Pi_1 & \Pi_2 \\ \dfrac{C}{D}\,\mathrm{R} & \dfrac{C}{D}\,\mathrm{R} \\ A \vee B\end{array}}{D}\vee\mathrm{E}(i,j) \tag{PC}$$

where R indicates any possible *elimination* rule of the calculus of which C occupies the position of major premiss. We will see in a moment that this restriction is crucial.

With this new reduction, the example above now does contain a redundancy, formed by $\vee$E followed by $\to$E, and thus reduces to the derivation

$$\frac{\overset{0}{A \vee A} \qquad \dfrac{\dfrac{\overset{1}{A}}{C \to A}\to\mathrm{I}(3) \qquad \overset{4}{C}}{A}\to\mathrm{E} \qquad \dfrac{\dfrac{\overset{2}{A}}{C \to A}\to\mathrm{I}(3) \qquad \overset{4}{C}}{A}\to\mathrm{E}}{A}\vee\mathrm{E}(1,2)$$

which now contains two more redundancies, and finally reduces to

$$\frac{\overset{0}{A \vee A} \qquad \overset{1}{A} \qquad \overset{2}{A}}{A}\vee\mathrm{E}(1,2)\cdot$$

Permutative conversions can be defined also for the false, via

$$\dfrac{\dfrac{\mathbf{F}}{C}\,\mathbf{F}\mathrm{E}}{D}\,\mathrm{R} \qquad \rightsquigarrow \qquad \dfrac{\mathbf{F}}{D}\,\mathbf{F}\mathrm{E}$$

Now, Prawitz proved that existence and unicity of normal forms extends to reductions enriched with permutative conversions, and that normal forms do satisfy the sub-formula property.

Theorem 6 (cf. [5])**.** **NJ***, with permutative conversions, for any derivation* Π *there exists a* unique *normal derivation* $\mathrm{nf}(\Pi)$*, such that* $\Pi \rightsquigarrow^* \mathrm{nf}(\Pi)$*. Moreover,* $\mathrm{nf}(\Pi)$ *enjoys the subformula property.*

Beyond Permutative Conversions The restriction to permutations with elimination rules might seem ad-hoc. Yet, it is enough to grant the subformula property. At the same time, it is unsatisfying both conceptually and semantically:

- conceptually, it is not clear why a permutation of an elimination rule should not change the proof, while a permutation of an introduction rule should, like e.g.

$$\dfrac{A \vee B \qquad \overset{i}{\underset{\Pi_1}{\overset{A}{C}}} \qquad \overset{j}{\underset{\Pi_2}{\overset{B}{C}}}}{\dfrac{C}{C \wedge D}\,\wedge\mathrm{I} \qquad D}\,\vee\mathrm{E}(i,j) \qquad \rightsquigarrow \qquad \dfrac{A \vee B \qquad \dfrac{\overset{i}{\underset{\Pi_1}{\overset{A}{C}}}}{C \wedge D}\,\wedge\mathrm{I} \qquad \dfrac{\overset{j}{\underset{\Pi_2}{\overset{B}{C}}}}{C \wedge D}\,\wedge\mathrm{I}}{C \wedge D}\,\vee\mathrm{E}(i,j)$$

  Observe how this example dangerously looks like the problematic conversion in MALL seen yesterday, the one responsible for an exponential size explosion:

$$\dfrac{\dfrac{\vdash \Gamma, C, A \qquad \vdash \Gamma, C, B}{\vdash \Gamma, C, A \& B}\,\&}{\vdash \Gamma, C \otimes D, A \& B}\,\otimes \qquad \sim \qquad \dfrac{\dfrac{\vdash \Gamma, C, A \qquad \vdash D}{\vdash \Gamma, C \otimes D, A}\,\otimes \qquad \dfrac{\vdash \Gamma, C, B \qquad \vdash D}{\vdash \Gamma, C \otimes D, B}\,\otimes}{\vdash \Gamma, C \otimes D, A \& B}\,\&$$

  Indeed, via de Morgan duality, one can transform the latter into:

$$\dfrac{\dfrac{A \vdash \Gamma, C \qquad \vdash \Gamma, C}{A \oplus B \vdash \Gamma, C}\,\oplus\mathrm{L} \qquad \vdash D}{A \oplus B \vdash \Gamma, C \otimes D}\,\otimes \qquad \sim \qquad \dfrac{\dfrac{A \vdash \Gamma, C \qquad \vdash D}{A \vdash \Gamma, C \otimes D}\,\otimes \qquad \dfrac{B \vdash \Gamma, C \qquad \vdash D}{B \vdash \Gamma, C \otimes D}\,\otimes}{A \oplus B \vdash \Gamma, C \otimes D}\,\oplus\mathrm{L}$$

  which is really alike the generalized conversion in natural deduction.
- Semantically, one can see that the interpretation of **NJ**-derivations in the appropriate family of models (called *bicartesian closed categories* and corresponding to cartesian closed categories having finite coproducts, i.e. disjunctions) does indeed validate *all* possible permutations with a $\vee$E.

One could thus try to bite the bullet, and consider stronger permutative rules, defined like (PC), but with R now indicating an *arbitrary* rule.

At first, this seems the right thing to do, because of the following result:

Theorem 7 (cf. [3])**.** *Let* $\equiv_{\mathrm{GR}}$ *indicate the reflexive, transitive* and symmetric *closure of* $\rightsquigarrow$ *with generalized permutative rules. Then, for any two derivations* Π, Π' *of* **NJ***,*

$$\Big(\textit{For all bicartesian closed category } \mathcal{C},\ [\![\Pi]\!]_{\mathcal{C}} = [\![\Pi']\!]_{\mathcal{C}}\Big) \quad \textit{iff} \quad \Pi \equiv_{\mathrm{GPC}} \Pi'.$$

In other words, generalized permutative conversions, together with usual reductions and expansions, precisely capture the equivalence classes of derivations obtained from categorical semantics. However, it is not by chance that we formulate the theorem above in terms of the equivalence $\equiv_{\mathrm{GR}}$ rather than in terms of normal forms. Indeed, such normal forms need not exist!

Proposition 8 (cf. [4])**.** *In presence of generalized permutative rules, reduction is non-terminating.*

Proof.

$$\Pi \quad = \quad \dfrac{A \vee B \qquad C \qquad \dfrac{A \vee B \qquad C \qquad C}{C}\,\vee\mathrm{E}}{C}\,\vee\mathrm{E}$$

reduces to

$$\dfrac{A \vee B \qquad \dfrac{A \vee B \qquad C \qquad C}{C}\,\vee\mathrm{E} \qquad \dfrac{A \vee B \qquad C \qquad C}{C}\,\vee\mathrm{E}}{C}\,\vee\mathrm{E}$$

which in turn reduces to

$$\dfrac{A \vee B \qquad \overset{\Pi}{C} \qquad \overset{\Pi}{C}}{C}\,\vee\mathrm{E}.$$

□

Observe that, since reduction is non-terminating, it is not *a priori* clear whether the equivalence $\equiv_{\mathsf{GPC}}$ is even decidable! Indeed, this problem remained open for many years, and was solved in 2017:

Theorem 9 (cf. [6])**.** $\equiv_{\mathsf{GPC}}$ *is decidable and coincides with* contextual equivalence *(something we will discuss on Friday).*

References

[1] Kosta Dosen. The Typed Böhm Theorem. *Electronic Notes in Theoretical Computer Science*, 50(2), 2001.

[2] Harvey Friedman. Equality between functionals. In *Logic Colloquium*, volume 453 of *Lecture Notes in Mathematics*, Berlin, Heidelberg, 1975. Springer.

[3] Joachim Lambek and Philip J. Scott. *Introduction to higher order categorical logic.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1988.

[4] Sam Lindley. Extensional rewriting with sums. In *Typed Lambda Calculi and Applications, TLCA 2007*, volume 4583 of *Lecture Notes in Computer Science*, pages 255–271. Springer Berlin Heidelberg, 2007.

[5] Dag Prawitz. Ideas and results in proof theory. In J.E. Fenstad, editor, *Proceedings of the 2nd Scandinavian Logic Symposium (Oslo)*, volume 63 of *Studies in logic and foundations of mathematics*. North-Holland, 1971.

[6] Gabriel Scherer. Deciding equivalence with sums and the empty type. In *Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages*, POPL 2017, pages 374–386, New York, NY, USA, 2017. ACM.

[7] Richard Statman. λ-definable functionals and $\beta\eta$-conversion. *Archiv für mathematische Logic und Grundlagenforschung*, 23:21–26, 1983.

[8] Richard Statman. Equality between functionals revisited. In *Harvey Friedman's research on the foundations of mathematics*, volume 117 of *Studies in Logic and Foundations of Mathematics*, pages 331–338. North-Holland, 1985.