Course Notes
“An Introduction to Proof Equivalence”

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Week 2 - Day 2

1 From Paradise to Hell

1.1 The Paradise of MLL

We consider the set of MLL-***fomulas*** generated from a set of atoms $\mathcal{A}$ using the following syntax.

$$A, B \quad \coloneqq \quad a \mid a^{\perp} \mid A \parr B \mid A \otimes B \mid \qquad \text{with } a \in \mathcal{A} \tag{1}$$

We consider formulas up to the following ***de Morgan*** laws:

$$(A^{\perp})^{\perp} = A \qquad (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp} (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$$

We define the ***linear implication*** $A \multimap B \coloneqq A^{\perp} \parr B$. A ***sequent*** is a set of (distinct) occurrences of formulas.

Example 1. The *barbara* syllogism “if A implies B and B implies C, then A implies C” can be represented using the following MLL-fomula:

$$((A \multimap B) \otimes (B \multimap C)) \multimap (A \multimap C) = ((A \otimes B^{\perp}) \parr (B \otimes C^{\perp})) \parr (A^{\perp} \parr C)$$

The ***multiplicative linear logic*** is defined by the proof system $\mathsf{MLL} = \{\mathsf{Ax}, \parr, \otimes\}$ defined by the rules in Figure 1 (the rule cut is admissible, therefore we do not consider it our definition of MLL).

Example 2. Consider the sequent $\Gamma \coloneqq A \otimes B^{\perp}, B \otimes C^{\perp}, C \parr A^{\perp}$. It admits only the two two following derivations in MLL.

$$\parr\dfrac{\otimes\dfrac{\mathsf{Ax}\dfrac{}{A, A^{\perp}} \qquad \otimes\dfrac{\mathsf{Ax}\dfrac{}{B, B^{\perp}} \quad \mathsf{Ax}\dfrac{}{C, C^{\perp}}}{B^{\perp}, B \otimes C^{\perp}, C}}{A \otimes B^{\perp}, B \otimes C^{\perp}, C, A^{\perp}}}{A \otimes B^{\perp}, B \otimes C^{\perp}, C \parr A^{\perp}} \qquad \parr\dfrac{\otimes\dfrac{\otimes\dfrac{\mathsf{Ax}\dfrac{}{A, A^{\perp}} \quad \mathsf{Ax}\dfrac{}{B, B^{\perp}}}{A \otimes B^{\perp}, B, A^{\perp}} \qquad \mathsf{Ax}\dfrac{}{C, C^{\perp}}}{A \otimes B^{\perp}, B \otimes C^{\perp}, C, A^{\perp}}}{A \otimes B^{\perp}, B \otimes C^{\perp}, C \parr A^{\perp}}$$

$$\mathsf{Ax}\,\frac{}{A, A^{\perp}} \qquad \parr\,\frac{\Gamma, A, B}{\Gamma, A \parr B} \qquad \otimes\,\frac{\Gamma, A \quad B, \Delta}{\Gamma, \Delta} \qquad \Bigg| \qquad \mathsf{cut}\,\frac{\Gamma, A \quad A^{\perp}, \Delta}{\Gamma, \Delta}$$

Figure 1: Sequent calculus rules for MLL and the rule cut.

$$\dfrac{\Gamma, A \quad \dfrac{\Delta, B, C, D}{\Delta, B, C \parr D}\parr}{\Gamma, \Delta, A \otimes B, C \parr D}\otimes = \otimes\dfrac{\otimes\dfrac{\Gamma, A \quad \Delta, B, C, D}{\Gamma, \Delta, \Sigma, A \otimes B, C, D}}{\Gamma, \Delta, A \otimes B, C \parr D} \qquad \parr\dfrac{\parr\dfrac{\Gamma, A, B, C, D}{\Gamma, A \parr B, C, D}}{\Gamma, A \parr B, C \parr D} = \parr\dfrac{\parr\dfrac{\Gamma, A, B, C, D}{\Gamma, A, B, C \parr D}}{\Gamma, A \parr B, C \parr D}$$

$$\dfrac{\Gamma, A \quad \dfrac{\Delta, B, C \quad \Sigma, D}{\Delta, \Sigma, B, C \otimes D}\otimes}{\Gamma, \Delta, \Sigma, A \otimes B, C \otimes D}\otimes = \otimes\dfrac{\otimes\dfrac{\Gamma, A \quad \Delta, B, C}{\Gamma, \Delta, \Sigma, A \otimes B, C} \quad \Sigma, D}{\Gamma, \Delta, \Sigma, A \otimes B, C \otimes D}$$

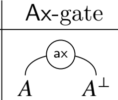
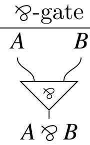
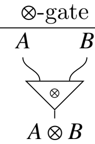
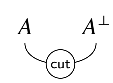
Figure 2: Rule permutations in MLL.

Note that sequent calculus derivations do not keep track of the complete order in which rules are applied: in the representation there is no record of the order in which we applied rules in two distinct branches. However, the formalism prevent us to similar irrelevant information about rules order as the order in which the two $\otimes$-rules are applied in the example above.

We consider the rule permutations in Figure 2 defined whenever two rules are applied to two distinct formulas of the sequent.

1.2 Getting rid of Bureaucracy

Definition 3. A MLL-proof structure is an hypergraph $S = \langle V, E \rangle$ whose ***vertices*** in V are labeled by MLL-fomulas (i.e., V is a set of occurrence of MLL-fomulas) and a whose (hyper)edges in E are called ***gates*** and respect the following local conditions on the labeling of the vertices:

Gate name	Ax-gate	$\parr$-gate	$\otimes$-gate	cut-gate
Shape				
Inputs	none	A and B	A and B	A and $A^{\perp}$
Outputs	A and $A^{\perp}$	$A \parr B$	$A \otimes B$	none

(2)

A vertex of a MLL-proof structure which is not the input of any of its gate is called an ***output*** or ***conclusion***.

A MLL-***proof net*** $P = [[\mathcal{D}]]_{\mathsf{MLL}}$ (with conclusion Γ) is a MLL-proof structure obtained from a derivation $\mathcal{D}$ (of a sequent Γ) in MLL using the translation inductively defined in Figure 3

Example 4. The left-hand side derivation in Example 2 has associated the following MLL-proof net.

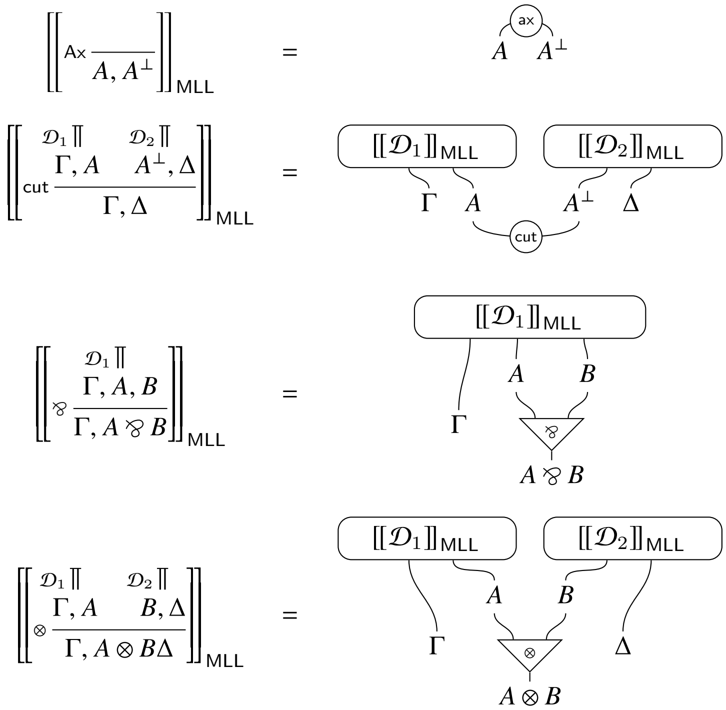
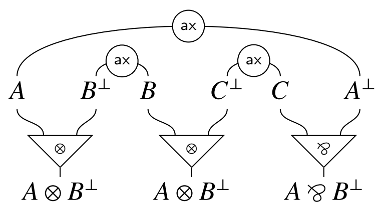
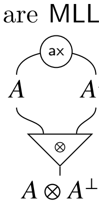


Figure 3: Translating a derivation in MLL into a MLL-proof net.



Remark 5. Since both connectives $\parr$ and $\otimes$ are binary, their gates have two inputs and one output. However, their rules have one and two premises respectively. As we will see later, this leads to a different behavior of these two type of gates when checking if a MLL-proof structure is a MLL-proof net.

Clearly, not all MLL-proof structures are MLL-proof nets. By means of example consider the MLL-proof structure  .

It is possible to define a topological characterization of those MLL-proof structures which are MLL-proof nets

Definition 6. Let S be a MLL-proof structure. A ***switching*** for S is a function σ selecting exactly one of the inputs of each $\parr$-gate. The ***test*** associated to a switching σ is the (hyper)graph $\sigma(S)$ defined by replacing each $\parr$-gate with an edge connecting its output with the input selected by σ.

The MLL-proof structure S is ***correct*** if the (hyper)graph $\sigma(S)$ is connected and acyclic for any possible switching σ.

Theorem 7. *Let S be a* MLL*-proof structure. Then S is a* MLL*-proof net iff S is correct.*

Proof. If S is a MLL-proof net, then there is a derivation $\mathcal{D}$ in MLL such that $S = [[\mathcal{D}]]_{\mathsf{MLL}}$. By the inductive definition of $[[\cdot]]_{\mathsf{MLL}}$, it suffices to check that the translation of each rule preserve correctness. In particular:

- Ax-rule is translated to a single Ax-gate, which is connected and acyclic;
- $\parr$-rule is translated into a $\parr$-gate connecting two outputs of a connected and acyclic MLL-proof net P_1. Therefore connectedness is preserved since the inputs of the new gate were already connected in P_1, and the output is connected to one of the two input in any possible test. Moreover, acyclicity is preserved since each possible switching never connect the two inputs of a same $\parr$-gate;
- $\otimes$-rule is translated into a $\otimes$-gate connecting two outputs of two distinct MLL-proof nets P_1 and P_2. The obtained MLL-proof structure is now connected via the new $\otimes$-gate. Moreover, it is acyclic since the existence of a cycle would imply the existence of two distinct paths connecting the inputs of the new $\otimes$-gate. This is impossible since all tests of P_1 and P_2 contain no cycles.

To prove the converse, we define a derivation $\mathcal{D}$ in MLL from a correct MLL-proof structure by induction on the number of gates of S with conclusion Γ.

- if S contains exactly one gate, then it must be a Ax-gate and $\mathcal{D}$ is made of a single Ax-rule;
- if S contains more than one gate then we consider the gates at the bottom of S. Then
  - if any such a gate is a $\parr$-gate, then, by definition, the MLL-proof structure S' defined by removing such a $\parr$-gate from S is still correct and contains strictly less gates than S. Then $\mathcal{D}$ is a derivation ending with a $\parr$-rule and whose premise is the conclusion of a derivation $\mathcal{D}'$ defined inductively from S';
  - or none of the gates at the bottom of S are $\parr$-gates. In this case, since S has more than one gate, there must be some $\otimes$-gates at the bottom of S. Moreover, the by removing any such a $\otimes$-gate we obtain two disconnected MLL-proof structures S_1 and S_2 which are both correct. Then $\mathcal{D}$ is a derivation ending with a $\otimes$-rule and whose premises are the conclusion of some derivation $\mathcal{D}_1$ and $\mathcal{D}_2$ defined inductively from S_1 and S_2 respectively. □

The interest in MLL-proof nets is that they are canonical with respect to the rule permutations in Figure 2.

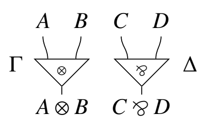
Proposition 8. *Using the definition provided here, check if a* MLL*-proof net is correct requires to check connectedness and acyclicity for* 2^k *tests where* k *is the number of* $\parr$*-Gates.*

However, more sophisticated correctness criterion allows us to check if a MLL*-proof net is correct in linear time with respect to the number of it gates [1].*

Theorem 9. *Let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be two derivations in* MLL*. Then it is possible to transform* $\mathcal{D}$ *into* $\mathcal{D}'$ *using the rule permutations in Figure 2 iff* $[\![\mathcal{D}]\!]_{\mathsf{MLL}} = [\![\mathcal{D}']\!]_{\mathsf{MLL}}$.

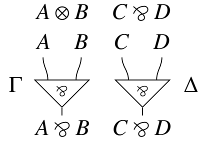
Proof. It suffices to prove that whenever a rule in Figure 2 is applied to a derivation in MLL, then the associated MLL-proof net does not change. For this purpose, it suffices to see that the MLL-proof structures associated to both sides of the equations defining the rule permutations are the same. The result follows by compositionality of $[\![\cdot]\!]_{\mathsf{MLL}}$.

$$\left[\!\!\left[\dfrac{\Gamma, A \quad \dfrac{B, C, D, \Delta}{B, C \parr D, \Delta}\parr}{\Gamma, A \otimes B, C \otimes D, \Delta}\otimes\right]\!\!\right]_{\mathsf{MLL}} =$$



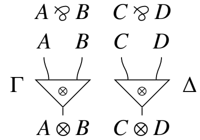
$$= \left[\!\!\left[\otimes\dfrac{\otimes\dfrac{\Gamma, A \quad B, C, D, \Delta}{\Gamma, \Sigma, A \otimes B, C, D, \Delta}}{\Gamma, A \otimes B, C \parr D, \Delta}\right]\!\!\right]_{\mathsf{MLL}}$$

$$\left[\!\!\left[\parr\dfrac{\parr\dfrac{\Gamma, A, B, C, D}{\Gamma, A \parr B, C, D}}{\Gamma, A \parr B, C \parr D}\right]\!\!\right]_{\mathsf{MLL}} =$$



$$= \left[\!\!\left[\parr\dfrac{\parr\dfrac{\Gamma, A, B, C, D}{\Gamma, A, B, C \parr D}}{\Gamma, A \parr B, C \parr D}\right]\!\!\right]_{\mathsf{MLL}}$$

$$\left[\!\!\left[\dfrac{\Gamma, A \quad \dfrac{\Delta, B, C \quad \Sigma, D}{\Delta, \Sigma, B, C \otimes D}\otimes}{\Gamma, \Delta, \Sigma, A \otimes B, C \otimes D}\otimes\right]\!\!\right]_{\mathsf{MLL}} =$$



$$= \left[\!\!\left[\otimes\dfrac{\otimes\dfrac{\Gamma, A \quad \Delta, B, C}{\Gamma, \Delta, \Sigma, A \otimes B, C} \quad \Sigma, D}{\Gamma, \Delta, \Sigma, A \otimes B, C \otimes D}\right]\!\!\right]_{\mathsf{MLL}}$$

□

Notation 10. Any MLL-proof net can be seen as the formula tree of its conclusion plus a decoration of its formulas linking the subformulas introduced by a same Ax-link.

For this purpose, from now on, we provide a more concise representation of cut-free MLL-proof nets by simply decorating a sequent Γ with some ***linkings*** pairing of subformulas in Γ which are paired by the Ax-rules. By means of example, the MLL-proof net in Example 4 can be represented as shown below.

$$A \otimes B^\perp, B \otimes C^\perp, C \parr A^\perp$$

1.3 The Forbidden Fruit(s)

1.3.1 Units

What happen if we include logical constants (i.e., units) in our syntax?

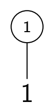
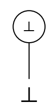
We now consider MLL-fomulas extended with the units $\bot$ and $\bot^\perp = 1$.

$$A, B \quad \coloneqq \quad a \mid a^\perp \mid A \parr B \mid A \otimes B \mid \bot \mid 1 \qquad \text{with } a \in \mathcal{A} \tag{3}$$

The system $\mathsf{MLL_u} = \mathsf{MLL} \cup \{\bot, 1\}$ is obtained by extending MLL with the two following rules:

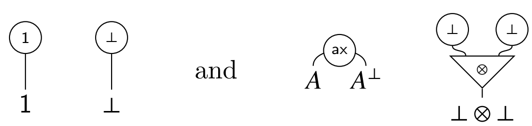
$$\bot\,\frac{\Gamma}{\Gamma, \bot} \qquad \text{and} \qquad 1\,\frac{}{1} \tag{4}$$

Intuitively, since both rules introduce a new constant in a sequent, they should both be represented by a gate with no inputs an one output. That is

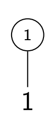
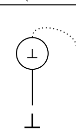
Gate name	1-gate	naïve $\bot$-gate
Shape		
Inputs	none	none
Outputs	1	$\bot$

However such a naïve encoding prevent us to have an efficient way to reconstruct a recognize those derivations in $\mathsf{MLL_u}$ which are proof nets (i.e., represent the encoding of a correct derivation).

Example 11. An example of two proof nets in $\mathsf{MLL_u}$ which are both disconnected. However, the one on the left correspond to a correct derivation while the one on the right cannot.



In order to recover the possibility of having an efficient correctness criterion (i.e., the possibility of checking in polynomial time w.r.t. the size of the proof structure whether it is a proof net), we need to include additional information in the encoding. This is included by the so-called ***jumps***, some additional edges connecting a $\bot$-gate to another gate of the MLL-proof structure [3].

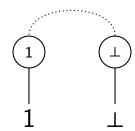
Gate name	1-gate	$\bot$-gate (with jump)
Shape		
Inputs	none	none
Outputs	1	$\bot$

(5)

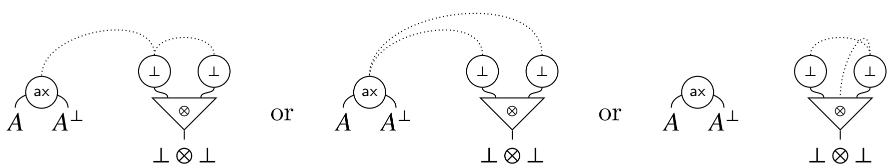
With those additional edges, we recover a correctness criterion based connectness and acyclicity of the tests (note that switching does not affect jumps)[1].

Definition 12. A $\mathsf{MLL_u}$-***proof net*** is a proof structure S with gates from Equations (2) and (5) such that all possible tests are connected and acyclic.

Example 13. The two $\mathsf{MLL_u}$-proof structures from Example 11 with additional jumps. The first one admits a unique possible way to assign jumps.



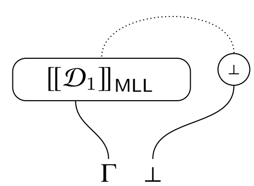
However, for the other we have $(\#\bot\text{-Gates}) \times (\#\text{Gates} - 1)$ possible way to assign each jumps of the 2 $\bot$-gates to one of the $(4-1)$ other gates. By means of example, we have the following possible proof structures with jumps:



We can extend the notation from Notation 10, depicting jumps as dotted links connecting an occurrence of $\bot$ to a link (representing a jump connecting the $\bot$-gate to a Ax-gate), to an occurrence of 1 (representing a jump connecting the $\bot$-gate to an 1-gate), or to an occurrence of a connective (representing a jump connecting the $\bot$-gate to the gate introducing the corresponding connective).

Definition 14. The translation $[\![\cdot]\!]_{\mathsf{MLL_u}}$ is defined extending the definition the one provided in Figure 3 by considering the two following cases for the rules 1 and $\bot$.

$$\left[\!\!\left[1\,\frac{}{1} \right]\!\!\right]_{\mathsf{MLL_u}} = 1 \qquad \left[\!\!\left[\begin{array}{c} \mathcal{D}_1 \,\| \\ \bot\,\dfrac{\Gamma}{\Gamma, \bot} \end{array} \right]\!\!\right]_{\mathsf{MLL}} = \tag{6}$$



where the jump of the $\bot$-gate is connected to the gate corresponding to the bottom-most rule in $\mathcal{D}_1$.

Example 15. The four proof nets in Example 13 can be respectively represented as shown below.

$$1 \quad \bot \qquad\quad A, A^{\perp}, \bot \otimes \bot \qquad\quad A, A^{\perp}, \bot \otimes \bot \qquad\quad A, A^{\perp}, \bot \otimes \bot$$

[1]Because of cut-elimination, it is better to assign jumps to Ax- and 1-rules.

$$\oplus\frac{\Gamma, A}{\Gamma, A \oplus B} \qquad \oplus\frac{\Gamma, A}{\Gamma, B \oplus A} \qquad \otimes\frac{\Gamma, A \quad B, \Gamma}{\Gamma, \Delta}$$

Figure 4: Sequent calculus rules for the additive connectives $\oplus$ and $\&$.

In [2] it has been shown that it is possible to use $\mathsf{MLL_u}$-proof nets containing only $\mathsf{1}$-, $\bot$-, $\parr$- and $\otimes$-gates to encode configurations of the constraint logic machine (a directed weighed graph), and instances of the *configuration-to-configuration* problem for these machines can be translated in instances of proof equivalence for $\mathsf{MLL_u}$-proof nets

Theorem 16. *Proof equivalence in* $\mathsf{MLL_u}$ *is* $\mathsf{P\text{-}Space}$*.*

1.3.2 Non-linear rules

What happen if we include rules for logical connectives which are not context-free?

We now consider MALL-fomulas extending the syntax of MLL-fomulas with the *additive connectives* for disjunction ($\oplus$) and conjunction ($\&$). More precisely:

$$\begin{array}{rcl} A, B & \coloneqq & a \mid a^\perp \mid A \parr B \mid A \otimes B \mid \\ & & \mid A \oplus B \mid A \,\&\, B \mid \end{array} \qquad \text{with } a \in \mathcal{A} \tag{7}$$

The ***multiplicative and additive linear logic*** is defined by the proof system $\mathsf{MALL} = \{\mathsf{Ax}, \parr, \otimes, \oplus, \&\}$ is defined by the rules in Figures 1 and 4.

When considering the possible independent rules in MALL in the style of the rule permutation in Figure 2, we observe that the permutation involving the $\&$-rule would duplicate some rules and sub-derivations.

As for MLL we have proof nets for MALL. To simplify the presentation, we refer to the formalism for MLL-proof nets introduced in Notation 10.

Definition 17. Let Γ be a sequent of MALL-fomulas. A ***$\&$-resolution*** (resp. a ***additive resolution***) of Γ is a sequent Γ^* obtained by removing the main connective and exactly one of the two subformulas of each $\&$-subformula (resp. $\oplus$- or $\&$-subformula) in Γ.

Example 18. Consider the following sequents and certain of their possible additive resolutions.

Formula	$\&$-Resolution	Additive Resolution
$A \,\&(B \parr)$	A (or $B \parr C$)	A (or $B \parr C$)
$C \oplus D$	$C \oplus D$	C (or D)
$(A \,\&(B \parr C)) \otimes ((D \oplus (E \parr F)) \parr G)$	$A \otimes ((D \oplus (E \parr F)) \parr G)$ or $(B \parr C) \otimes ((D \oplus (E \parr F)) \parr G)$	$A \otimes ((E \parr F) \parr G)$ or $(B \parr C) \otimes (D \parr G)$

Definition 19 ([5])**.** Let Γ be a sequent of MALL-fomulas. A ***linking*** λ for Γ is a set of pairs of complementary literals in Γ (i.e., an atom and its dual atom).

$$\frac{\Gamma, A \quad \dfrac{\Delta, B, C, D}{\Delta, B, C \parr D}\parr}{\Gamma, \Delta, A \otimes B, C \parr D}\otimes = \frac{\dfrac{\Gamma, A \quad \Delta, B, C, D}{\Gamma, \Delta, \Sigma, A \otimes B, C, D}\otimes}{\Gamma, \Delta, A \otimes B, C \parr D}\parr \qquad \frac{\dfrac{\Gamma, A, B, C, D}{\Gamma, A \parr B, C, D}\parr}{\Gamma, A \parr B, C \parr D}\parr = \frac{\dfrac{\Gamma, A, B, C, D}{\Gamma, A, B, C \parr D}\parr}{\Gamma, A \parr B, C \parr D}\parr$$

$$\frac{\Gamma, A \quad \dfrac{\Delta, B, C \quad \Sigma, D}{\Delta, \Sigma, B, C \otimes D}\otimes}{\Gamma, \Delta, \Sigma, A \otimes B, C \otimes D}\otimes = \frac{\dfrac{\Gamma, A \quad \Delta, B, C}{\Gamma, \Delta, \Sigma, A \otimes B, C}\otimes \quad \Sigma, D}{\Gamma, \Delta, \Sigma, A \otimes B, C \otimes D}\otimes$$

$$\frac{\Gamma, A \quad \dfrac{\Delta, B, C}{\Delta, B, C \oplus D}\oplus}{\Gamma, \Delta, A \otimes B, C \oplus D}\otimes = \frac{\dfrac{\Gamma, A \quad \Delta, B, C}{\Gamma, \Delta, \Sigma, A \otimes B, C}\otimes}{\Gamma, \Delta, A \otimes B, C \oplus D}\oplus \qquad \frac{\dfrac{\Gamma, A, B, C}{\Gamma, A \parr B, C}\parr}{\Gamma, A \oplus B, C \oplus D}\oplus = \frac{\dfrac{\Gamma, A, B, C}{\Gamma, A, B, C \oplus D}\oplus}{\Gamma, A \parr B, C \oplus D}\parr$$

$$\frac{\dfrac{\Gamma, A, C \quad \Gamma, A, D}{\Gamma, A, C \with D}\with \quad \dfrac{\Gamma, B, C \quad \Gamma, B, D}{\Gamma, B, C \otimes D}\with}{\Gamma, A \with B, C \with D}\with = \frac{\dfrac{\Gamma, A, C \quad \Gamma, B, C}{\Gamma, A \with B, C}\with \quad \dfrac{\Gamma, A, D \quad \Gamma, B, D}{\Gamma A \with B, D}\with}{\Gamma, A \with B, C \with D}\with$$

$$\frac{\Gamma, A \quad \dfrac{\Gamma, B, C}{\Gamma, B, C \oplus D}\oplus}{\Gamma, A \otimes B, C \oplus D}\with = \frac{\dfrac{\Gamma, A \quad \Delta, B, C}{\Gamma, \Delta, \Sigma, A \otimes B, C}\with}{\Gamma, \Delta, A \otimes B, C \oplus D}\oplus \qquad \frac{\dfrac{\Gamma, A, C \quad \Gamma, B, C}{\Gamma, A \with B, C}\with}{\Gamma, A \with B, C \oplus D}\oplus = \frac{\dfrac{\Gamma, A, C}{\Gamma, A, C \oplus D}\oplus \quad \dfrac{\Gamma, B, C}{\Gamma, B, C \oplus D}\oplus}{\Gamma, A \with B, C \oplus D}\with$$

$$\frac{\Gamma, A \quad \dfrac{\Delta, B, C \quad \Delta, B, D}{\Delta, B, C \with D}\with}{\Gamma, \Delta, A \otimes B, C \with D}\otimes = \frac{\dfrac{\Gamma, A \quad \Delta, B, C}{\Gamma, \Delta, A \otimes B, C}\otimes \quad \dfrac{\Gamma, A \quad \Delta, B, C}{\Gamma, \Delta, A \otimes B, C}\otimes}{\Gamma, \Delta, A \otimes B, C \with D}\with$$

Figure 5: Rule permutations in MALL.

A linking is ***over*** an additive resolution Γ^* of Γ if each occurrence of a literal in λ occurs in Γ^*.

A ***(*MALL*) slice net*** for a sequent Γ is a set of linking Λ over Γ such that, for each $\with$-resolution Γ^* of Γ there is a unique linking $\lambda \in \Lambda$ over Γ^* and λ identifies a MLL-proof net for an additive resolution $\Gamma^\star$.

Given a derivation in MALL, we can defined a slice net enconding it using the translation in Figure 6

Proposition 20. *It is not possible to check if a set of linking* Γ *over a sequent* Γ *is a slice net in* P-Time*. More precisely, it requires an exponential number* 2^k *where* k *is the number of occurrences of the connective* $\with$ *in* Γ*.*

As for MLL-proof nets, we have a canonicity result for slice nets.

Theorem 21. *Let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be two derivations in* MALL*. Then it is possible to transform* $\mathcal{D}$ *into* $\mathcal{D}'$ *using the rule permutations in Figure 5 iff* $[[\mathcal{D}]]_{\mathsf{Slice}} = [[\mathcal{D}']]_{\mathsf{Slice}}$.

Corollary 22. *Compute the conflict net associate to a derivation in* MALL *requires an exponential number of steps.*

$$\left[\!\!\left[\mathsf{ax}\,\frac{}{a, a^{\perp}} \right]\!\!\right]_{\mathsf{Slice}} = \{\{a, a^{\perp}\}\}$$

$$\left[\!\!\left[\parr\,\frac{\begin{array}{c}\mathcal{D}_1 \,\| \\ \Gamma, A, B\end{array}}{\Gamma, A \parr B} \right]\!\!\right]_{\mathsf{Slice}} = \left[\!\!\left[\oplus\,\frac{\begin{array}{c}\mathcal{D}_1 \,\| \\ \Gamma, A\end{array}}{\Gamma, A \oplus B} \right]\!\!\right]_{\mathsf{Slice}} = \left[\!\!\left[\oplus\,\frac{\begin{array}{c}\mathcal{D}_1 \,\| \\ \Gamma, A\end{array}}{\Gamma, B \oplus A} \right]\!\!\right]_{\mathsf{Slice}} = [\![\mathcal{D}_1]\!]_{\mathsf{Slice}}$$

$$\left[\!\!\left[\otimes\,\frac{\begin{array}{cc}\mathcal{D}_1 \,\| & \mathcal{D}_2 \,\| \\ \Gamma, A & B, \Delta\end{array}}{\Gamma, A \otimes B \Delta} \right]\!\!\right]_{\mathsf{Slice}} = \{\lambda_1 \cup \lambda_2 \mid \lambda_1 \in [\![\mathcal{D}_1]\!]_{\mathsf{Slice}} \quad \text{and} \quad \lambda_2 \in [\![\mathcal{D}_2]\!]_{\mathsf{Slice}}\}$$

$$\left[\!\!\left[\&\,\frac{\begin{array}{cc}\mathcal{D}_1 \,\| & \mathcal{D}_2 \,\| \\ \Gamma, A & \Gamma, B\end{array}}{\Gamma, A \& B} \right]\!\!\right]_{\mathsf{Slice}} = [\![\mathcal{D}_1]\!]_{\mathsf{Slice}} \cup [\![\mathcal{D}_2]\!]_{\mathsf{Slice}}$$

Figure 6: Translating a derivation in MALL into a slice nets.

Theorem 23. *Let $\mathcal{D}$ and $\mathcal{D}'$ be two derivations in* MALL*. Then it is possible to transform $\mathcal{D}$ into $\mathcal{D}'$ using the rule permutations in Figure 5 iff* $[\![\mathcal{D}]\!]_{\mathsf{Slice}} = [\![\mathcal{D}']\!]_{\mathsf{Slice}}$.

We do not enter in the details here, but in conflict nets it is possible to represent cuts without affecting their efficiency w.r.t. translation. Moreover, cut-elimination can be performed in polynomial time.

1.3.3 Conflict Nets

It is possible to provide a syntax for MALL-proof nets with an efficient proof translation, but we show that such a result requires to weaken the proof equivalence captured by the syntax (and preventing an efficient procedure for cut-elimination).

Definition 24. Let Γ be a sequent (of MALL-fomulas). A ***link*** on a sequent Γ is a sub-sequent of Γ such that each each occurrence of a subformula of Γ can occur in at most one formula in λ.

A ***cotree*** on Γ is a tree of $\#$- and $\frown$-nodes whose leaves links on Γ. It is ***axiomatic*** if all links are pairs of formulas of the form $a, a^{\perp}$ with $a \in \mathcal{A}$. It is in ***alternating form*** if it contains no internal nodes with a unique child, and if it contains no two adjacent nodes which are both $\#$-nodes or $\frown$-nodes.

Example 25. Consider the sequent $\Gamma = A \& B, B^{\perp} \otimes C^{\perp}, C \oplus D$. Then we have

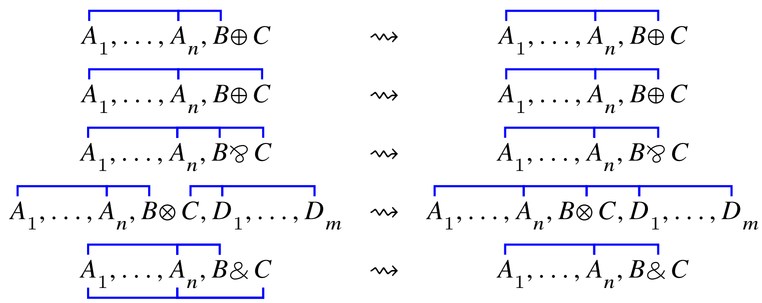
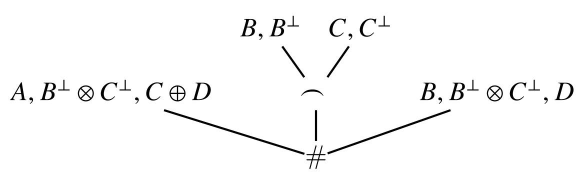


Figure 7: Coalescence rules for cotrees (we omitted rules normalizing cotrees in alternating forms).

the following links:

Link	Link representation
$A, B^\perp \otimes C^\perp, C \oplus D$	$A \& B, B^\perp \otimes C^\perp, C \oplus D$
$B, B^\perp$	$A \& B, B^\perp \otimes C^\perp, C \oplus D$
$C^\perp, C$	$A \& B, B^\perp \otimes C^\perp, C \oplus D$
$B, B^\perp \otimes C^\perp, D$	$A \& B, B^\perp \otimes C^\perp, C \oplus D$

The following is a cotree over Γ.



which can be also be represented in-line as follows:

$$\#\left(A, B^\perp \otimes C^\perp, C \oplus D; \frown \left(B, B^\perp; C, C^\perp\right); B, B^\perp \otimes C^\perp, D;\right)$$

It is possible to translate (in $\mathsf{P\text{-}Time}$) a derivation in MALL into an axiomatic linking using the translation in Figure 8.

Definition 26 ([4])**.** A ***conflict net*** (for Γ) is a cotree Λ on Γ such that $\Lambda \rightsquigarrow \Gamma$ via the rules in Figure 7.

Proposition 27. *It is possible to check in* $\mathsf{P\text{-}Time}$ *if a cotree* Λ *is a conflict net on a sequent* Γ*.*

Theorem 28. *Let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be two derivations in* MALL*. Then it is possible to transform* $\mathcal{D}$ *into* $\mathcal{D}'$ *using all local rule permutations in Figure 5 (i.e., all rules except the bottom-most one) iff* $[[\mathcal{D}]]_{\mathsf{Conflict}} = [[\mathcal{D}']]_{\mathsf{Conflict}}$.

$$\left[\!\!\left[\mathsf{ax}\,\frac{}{a, a^{\perp}} \right]\!\!\right]_{\mathsf{Conflict}} = \{\{a, a^{\perp}\}\}$$

$$\left[\!\!\left[\mathcal{D}_1 \Vert \atop \text{⅋}\,\frac{\Gamma, A, B}{\Gamma, A \text{⅋} B} \right]\!\!\right]_{\mathsf{Conflict}} = \left[\!\!\left[\mathcal{D}_1 \Vert \atop \oplus\,\frac{\Gamma, A}{\Gamma, A \oplus B} \right]\!\!\right]_{\mathsf{Conflict}} = \left[\!\!\left[\mathcal{D}_1 \Vert \atop \oplus\,\frac{\Gamma, A}{\Gamma, B \oplus A} \right]\!\!\right]_{\mathsf{Conflict}} = [\![\mathcal{D}_1]\!]_{\mathsf{Conflict}}$$

$$\left[\!\!\left[\mathcal{D}_1 \Vert \quad \mathcal{D}_2 \Vert \atop \otimes\,\frac{\Gamma, A \qquad B, \Delta}{\Gamma, A \otimes B \Delta} \right]\!\!\right]_{\mathsf{Conflict}} = \frown \left([\![\mathcal{D}_1]\!]_{\mathsf{Conflict}} \,;\, [\![\mathcal{D}_2]\!]_{\mathsf{Conflict}}\right)$$

$$\left[\!\!\left[\mathcal{D}_1 \Vert \quad \mathcal{D}_2 \Vert \atop \&\,\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \,\&\, B} \right]\!\!\right]_{\mathsf{Conflict}} = \# \left([\![\mathcal{D}_1]\!]_{\mathsf{Conflict}} \,;\, [\![\mathcal{D}_2]\!]_{\mathsf{Conflict}}\right)$$

Figure 8: Translating a derivation in MALL into a slice nets.

	Proof system (i.e., check correctness in P-Time)	Canonical (i.e., $\mathcal{D} \simeq \mathcal{D}'$ iff $[\![\mathcal{D}]\!] = [\![\mathcal{D}']\!]$)
MLL	✓	✓
naïve $\mathsf{MLL_u}$	X	✓
$\mathsf{MLL_u}$ (with jumps)	✓	X
MALL slice nets	X	✓
MLL conflict nets	✓	✓ (without &-⊗ permutation)

Figure 9: Brief summary of the results for various proof nets from the literature discussed here.

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