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> LJUBLJANA > SLOVENIA

Introduction to Proof Equivalence

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Class 1: Proof Systems and Proof Equivalence

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Overview

Class 1:
- Overview on proof equivalence
- Proof systems: natural deduction and sequent calculus

Class 2:
- The paradyse of syllogism and the forbidden fruit
- Resource management in proofs

Class 3:
- Normalization-equivalence in intuitionistic logic
- Denotational semantics

Class 4:
- Permutation-equivalence in sequent calculus
- Combinatorial proofs

Class 5:
- Proof equivalence in first and second order logic
- Proof equivalence in computer science

Obvious topics in general proof theory are:

2.1. The basic question of defining the notion of proof, including the question of the distinction between different kinds of proofs such as constructive proofs and classical proofs.

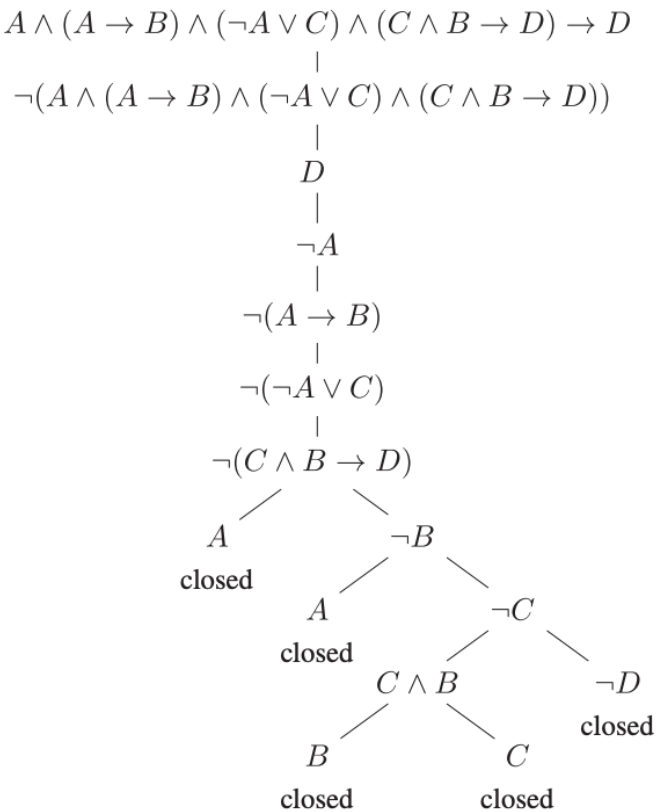
2.2. Investigation of the structure of (different kinds of) proofs, including e.g. questions concerning the existence of certain normal forms.

2.3. The representation of proofs by formal derivations. In the same way as one asks when two formulas define the same set or two sentences express the same proposition, one asks when two derivations represent the same proof; in other words, one asks for identity criteria for proofs or for a "synonymity" (or equivalence) relation between derivations.

[Prawitz, *Ideas and results in proof theory*, 1971]

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax \quad \dfrac{}{B \vdash B}\,ax}{A, B \vdash A \wedge B}\,\wedge\mathrm{R} \quad \dfrac{}{C \vdash C}\,ax}{A, C \to B, C \vdash A \wedge B}\,\to\mathrm{L}}{A, C \to B \vdash C \to (A \wedge B)}\,\to\mathrm{R}$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax \quad \dfrac{\dfrac{}{B \vdash B}\,ax \quad \dfrac{}{C \vdash C}\,ax}{C \to B, C \vdash B}\,\to\mathrm{L}}{A, C \to B, C \vdash A \wedge B}\,\wedge\mathrm{R}}{A, C \to B \vdash C \to (A \wedge B)}\,\to\mathrm{R}$$



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[F]}{\neg A \vee C} \quad \dfrac{\dfrac{\dfrac{[F]}{A} \quad [\neg A]}{\bot}}{\bot \quad \dfrac{\bot}{C}} \quad [C]}{\vee_E}}{C} \quad \dfrac{\dfrac{[F]}{A} \quad \dfrac{[F]}{A \to B}}{B}}{\wedge_I}}{C \wedge B} \quad \dfrac{[F]}{C \wedge B \to D}}{D}}{A \wedge (A \to B) \wedge (\neg A \vee C) \wedge (C \wedge B \to D) \to D}$$

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Goal A /\ (A -> B) /\ (~A \/ C) /\ (C /\ B -> D) -> D.
Proof.
intros h1. destruct h1 as [ha h2].
destruct h2 as [hab h3]. destruct h3 as [hac h4].
apply h4. split.
apply hab. exact ha.
destruct hac as [hna|hc]. elim hna. exact ha.
exact hc.
Qed.
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Introduction Rules

$$\frac{}{\mathbf{T}}\,\mathbf{T}\mathrm{I} \qquad \frac{A \qquad B}{A \wedge B}\,\wedge\mathrm{I} \qquad \frac{A_i}{A \vee B}\,\vee\mathrm{I}_i \quad (i \in \{0,1\}) \qquad \frac{\begin{array}{c}\overset{i}{[A]}\\ \vdots \\ B\end{array}}{A \to B}\,\to\mathrm{I}(i)$$

Elimination Rules

$$\frac{\mathbf{F}}{A}\,\mathbf{F}\mathrm{E} \qquad \frac{A_0 \wedge A_1}{A_i}\,\wedge\mathrm{E}_i \quad (i \in \{0,1\}) \qquad \frac{A \vee B \qquad \begin{array}{c}\overset{i}{A}\\ \vdots \\ C\end{array} \qquad \begin{array}{c}\overset{j}{B}\\ \vdots \\ C\end{array}}{C}\,\vee\mathrm{E}(i,j) \qquad \frac{A \to B \qquad A}{B}\,\to\mathrm{E}$$

$$\dfrac{\dfrac{\overset{0}{A}}{A \to A}\,{\to}\mathrm{I}(1)}{A \to (A \to A)}\,{\to}\mathrm{I}(0)$$

$$\dfrac{\dfrac{\overset{1}{A}}{A \to A}\,{\to}\mathrm{I}(1)}{A \to (A \to A)}\,{\to}\mathrm{I}(0)$$

$$\dfrac{\overset{0}{A}}{A \to A} \to \mathrm{I}(0) \qquad \dfrac{\overset{1}{A}}{A \to A} \to \mathrm{I}(1)$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ A\end{matrix} \quad \begin{matrix}\Pi_2 \\ B\end{matrix}}{A \wedge B}\wedge\mathrm{I}}{A}\wedge\mathrm{E}_1 \quad \rightsquigarrow \quad \begin{matrix}\Pi_1 \\ A\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\overset{i}{A} \\ \Pi_1 \\ B\end{matrix}}{A \to B}\to\mathrm{I}(i) \quad \begin{matrix}\Pi_2 \\ A\end{matrix}}{B}\to\mathrm{E} \quad \rightsquigarrow \quad \begin{matrix}\Pi_2 \\ [A] \\ \Pi_1 \\ B\end{matrix}$$

Identity Group

$$\frac{}{A \vdash A}\,\mathrm{ax} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{cut}$$

Structural Group

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,\mathrm{LW} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,\mathrm{LC} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\,\mathrm{RW} \qquad \frac{\Gamma \vdash, A, A\Delta}{\Gamma \vdash A, \Delta}\,\mathrm{RC}$$

Logical Group

$$\frac{}{\vdash \mathbf{T}}\,\mathrm{R}\mathbf{T} \qquad \frac{}{\mathbf{F} \vdash}\,\mathrm{L}\mathbf{F}$$

$$\frac{\Gamma, A_i \vdash \Delta}{\Gamma, A_0 \wedge A_1 \vdash \Delta}\,\mathrm{L}\wedge_i \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\,\mathrm{L}\vee \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\,\mathrm{R}\wedge \qquad \frac{\Gamma \vdash A_i, \Delta}{\Gamma \vdash A_0 \vee A_1, \Delta}\,\mathrm{R}\vee_i$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \to B \vdash \Delta, \Delta'}\,\mathrm{L}\to \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\,\mathrm{R}\to$$

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\Pi_1}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}\,\mathrm{R}\wedge \quad \dfrac{\dfrac{\Sigma}{\Gamma', A \vdash \Delta'}}{\Gamma', A \wedge B \vdash \Delta'}\,\mathrm{L}\wedge}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\Sigma}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{cut}$$

$$\Pi_1 = \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{A, B \vdash A}\,w \quad \dfrac{\dfrac{}{B \vdash B}\,ax}{A, B \vdash B}\,w}{A, B \vdash A \wedge B}\,\wedge R \quad \dfrac{}{C \vdash C}\,ax}{A, C \to B, C \vdash A \wedge B}\,\to L$$

$$\Pi_2 = \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{A, C \to B, C \vdash A}\,w \quad \dfrac{\dfrac{}{B \vdash B}\,ax \quad \dfrac{}{C \vdash C}\,ax}{C \to B, C \vdash B}\,\to L}{A, C \to B, C \vdash A \wedge B}\,\wedge R}$$

$$\dfrac{\overset{i}{A} \qquad \dfrac{C \to B \qquad \overset{k}{C}}{B}\,\to E}{A \wedge B}\,\wedge I$$

$$\dfrac{\Gamma \vdash A \qquad \dfrac{B, \Delta, C \vdash D}{B, \Delta \vdash C \to D}\,\mathrm{R}\to}{\Gamma, A \to B, \Delta \vdash C \to D}\,\mathrm{L}\to \qquad \sim_{\mathsf{p}} \qquad \dfrac{\dfrac{\Gamma \vdash A \qquad B, \Delta, C \vdash D}{\Gamma, A \to B, \Delta, C \vdash D}\,\mathrm{L}\to}{\Gamma, A \to B, \Delta \vdash C \to D}\,\mathrm{R}\to$$

$$\dfrac{\Gamma \vdash A \qquad \dfrac{B, \Delta, C \vdash D}{B, \Delta \vdash C \to D}\,\mathrm{R}\to}{\Gamma, A \to B, \Delta \vdash C \to D}\,\mathrm{L}\to \quad \sim_{\mathsf{p}} \quad \dfrac{\dfrac{\Gamma \vdash A \qquad B, \Delta, C \vdash D}{\Gamma, A \to B, \Delta, C \vdash D}\,\mathrm{L}\to}{\Gamma, A \to B, \Delta \vdash C \to D}\,\mathrm{R}\to$$

$$\dfrac{\dfrac{\dfrac{\Gamma, A, A, B, B \vdash C}{\Gamma, A, B, B \vdash C}\,\mathrm{LC}}{\Gamma, A, B \vdash C}\,\mathrm{LC}}{\Gamma, A \wedge B \vdash C}\,\mathrm{L}\wedge \quad \sim_{\mathsf{p}} \quad \dfrac{\dfrac{\dfrac{\Gamma, A, A, B, B \vdash C}{\Gamma, A \wedge B, A, B \vdash C}\,\mathrm{L}\wedge}{\Gamma, A \wedge B, A \wedge B \vdash C}\,\mathrm{L}\wedge}{\Gamma, A \wedge B \vdash C}\,\mathrm{LC}$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash C}{\Gamma, A \vdash C}\,\mathrm{LW}}{\Gamma, A, B \vdash C}\,\mathrm{LW}}{\Gamma, A \wedge B \vdash C}\,\mathrm{L}\wedge \quad \sim_{\mathsf{p}} \quad \dfrac{\Gamma \vdash C}{\Gamma, A \wedge B \vdash C}\,\mathrm{LW}$$

$$\dfrac{\dfrac{\Gamma, A \vdash B}{\Gamma, A, A \vdash C}\,\mathrm{LW}}{\Gamma, A \vdash C}\,\mathrm{LC} \quad \sim_{\mathsf{p}} \quad \Gamma, A \vdash B$$

$$\dfrac{\Gamma \vdash A \qquad \dfrac{B, \Delta, C \vdash D}{B, \Delta \vdash C \to D}\,\mathrm{R}\to}{\Gamma, A \to B, \Delta \vdash C \to D}\,\mathrm{L}\to \quad \sim_{\mathsf{p}} \quad \dfrac{\dfrac{\Gamma \vdash A \qquad B, \Delta, C \vdash D}{\Gamma, A \to B, \Delta, C \vdash D}\,\mathrm{L}\to}{\Gamma, A \to B, \Delta \vdash C \to D}\,\mathrm{R}\to$$

$$\dfrac{\dfrac{\dfrac{\Gamma, A, A, B, B \vdash C}{\Gamma, A, B, B \vdash C}\,\mathrm{LC}}{\Gamma, A, B \vdash C}\,\mathrm{LC}}{\Gamma, A \wedge B \vdash C}\,\mathrm{L}\wedge \quad \sim_{\mathsf{p}} \quad \dfrac{\dfrac{\dfrac{\Gamma, A, A, B, B \vdash C}{\Gamma, A \wedge B, A, B \vdash C}\,\mathrm{L}\wedge}{\Gamma, A \wedge B, A \wedge B \vdash C}\,\mathrm{L}\wedge}{\Gamma, A \wedge B \vdash C}\,\mathrm{LC}$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash C}{\Gamma, A \vdash C}\,\mathrm{LW}}{\Gamma, A, B \vdash C}\,\mathrm{LW}}{\Gamma, A \wedge B \vdash C}\,\mathrm{L}\wedge \quad \sim_{\mathsf{p}} \quad \dfrac{\Gamma \vdash C}{\Gamma, A \wedge B \vdash C}\,\mathrm{LW}$$

$$\dfrac{\dfrac{\dfrac{\Gamma, A \vdash B}{\Gamma, A, A \vdash C}\,\mathrm{LW}}{\Gamma, A \vdash C}\,\mathrm{LC} \quad \sim_{\mathsf{p}} \quad \Gamma, A \vdash B$$

$$\dfrac{\Gamma \vdash A \qquad \dfrac{B, B, \Delta \vdash C}{B, \Delta \vdash C}\,\mathrm{LC}}{\Gamma, A \to B, \Delta \vdash C}\,\mathrm{L}\to \quad \sim_{\mathsf{p}} \quad \dfrac{\Gamma \vdash A \qquad \dfrac{\Gamma \vdash A \qquad B, B, \Delta \vdash C}{B, \Gamma, A \to B, \Delta \vdash C}\,\mathrm{L}\to}{\Gamma, A \to B, A \to B, \Delta \vdash C}\,\mathrm{L}\to}{\Gamma, A \to B, \Delta \vdash C}\,\mathrm{LC}$$

$$\dfrac{\Gamma \vdash A \qquad \dfrac{\Delta \vdash C}{B, \Delta \vdash C}\,\mathrm{LW}}{\Gamma, A \to B, \Delta \vdash C}\,\mathrm{L}\to \quad \sim_{\mathsf{p}} \quad \dfrac{\Delta \vdash C}{\Gamma, A \to B, \Delta \vdash C}\,\mathrm{LW}$$